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Branes in the $2D$ black hole

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Abstract: We present a comprehensive analysis of branes in the euclidean $2D$ black hole (cigar). In particular, exact boundary states and annulus amplitudes are provided for D0-branes which are localized at the tip of the cigar as well as for two families of extended D1 and D2-branes. Our results are based on closely related studies for the euclidean AdS_3 model [1] and, as predicted by the conjectured duality between the $2D$ black hole and the sine-Liouville model, they share many features with branes in Liouville theory. New features arise here due to the presence of closed string modes which are localized near the tip of the cigar. The paper concludes with some remarks on possible applications to exact tachyon condensation and matrix models.

Keywords: D-branes, Conformal Field Models in String Theory.

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Contents

1. Introduction

There exists an interesting exact $2D$ classical solution of the string equations of motion which became known as the *cigar* and which has been studied for various different reasons after it was first discovered in [2, 3]. Initially, investigations focused mainly on a lorentzian version of the cigar geometry because is was observed to exhibit all characteristic features of a 1+1 dimensional black hole [4]. The euclidean counterpart, which we shall mostly refer to as $2D$ *black hole*, gained more relevance later as a building block for several important 9+1-dimensional string backgrounds [5, 6]. In particular, it arises when a stack of NS5-branes is separated along a 1-dimensional circle [7] in order to prevent the string coupling constant from diverging at the branes' locus [6]. Hence, a detailed study of strings and branes in the $2D$ black hole geometry can be seen as a crucial step toward understanding string theory in the near horizon geometry of NS5-branes and its conjectured duality with little string theory (see e.g. [8] and references therein).

The spectrum of closed string modes in the $2D$ black hole geometry was originally analyzed by Dijkgraaf and the Verlindes [9]. Even though their proposal was partly based on a conjectural and somewhat obscure duality with the so-called $2D$ trumpet geometry, it has been confirmed recently through an exact computation of the partition function [10].[1] What makes the whole structure of the spectrum non-trivial is the existence of closed string modes which are localized near the tip of the cigar. Other exact quantities in the bulk theory, in particular the closed string 3-point couplings, can be inferred from Teschner's work on the H_3 model [11, 12]. The latter is related to the $\mathrm{SL}(2,\mathbb{R})$ WZW model by a Wick rotation in target space and it serves as the starting point for a coset construction of the $2D$ black hole.

In this work we extend the exact solution of string theory in the $2D$ black hole background to branes and open strings. One obvious motivation comes from the duality with little string theory that we have mentioned before. It is also worth pointing out that the $2D$ black hole was conjectured by Fateev, Zamolodchikov and Zamolodchikov to be T-dual to sine-Liouville theory (see [13] for a more precise description of the conjecture). A supersymmetric version of this T-duality [14] which involves the $N = 2$ $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki quotient on one side and $N = 2$ Liouville theory on the other has been firmly established [15]. In this light, our studies of branes and open strings in the $2D$ black hole appear as a 2-dimensional generalization of similar investigations for Liouville theory [16, 17]. We shall find many parallels between branes in the two models.

Our construction of branes and their open string spectra in the $2D$ black hole background departs from the results of [1, 18] on D-branes in H_3 (see also [19]–[21] for some proposals and partial results on exact boundary theories for branes in H_3). In principle, these descend down from the 3-dimensional H_3 model through a rather simple coset procedure involving some 1-parameter family of translations. The implications for the 2-dimensional quotient, however, are quite non-trivial due to the discrete bulk modes that are localized near the tip of the $2D$ black hole. Note that these have no analogue in the bulk spectrum of the H_3 -model. Hence, e.g. the Cardy-type consistency conditions which arise from world-sheet duality differ significantly between the parent theory and its coset.

We shall start our presentation with a review of some relevant material on the bulk theory. This is followed by three rather independent sections on three different types of branes. The latter may be classified according to their dimension and they include a set of point-like branes as well as two families of extended D1 and D2-branes. Each section below contains a careful study of the semi-classical limit in which we describe the branes'

[1] There is a only a small discrepancy concerning the so-called unitarity bound.

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geometry and some properties of their open string spectra.[2] A comprehensive summary of the exact solutions is provided and motivated from the results of [1] before we explain how to verify the consistency with world-sheet duality.[3] The paper concludes with a list of open problems.

2. Preliminaries: closed strings on the cigar

To prepare for our study of branes and open strings in the cigar geometry we will need to review some background material about the closed string theory. We will start with a few words on the minisuperspace model, partly to motivate the description of the spectrum in the full conformal field theory which we address in the second subsection. The section concludes with a short review of the coset construction from H_3 .

2.1 The minisuperspace model

The 2-dimensional cigar is parametrized by some angle $\theta \in [0, 2\pi[$ and a radial coordinate $\rho \in [0, \infty[$ with $\rho = 0$ corresponding to the tip (see figure 1). In these coordinates, the background metric and dilaton are given by the expressions

$$ds^2 = \frac{k}{2}\,(d\rho^2 + \tanh^2\rho\, d\theta^2)\,, \qquad e^{\Phi} = \frac{e^{\Phi_0}}{\cosh\rho}\,, \tag{2.1}$$

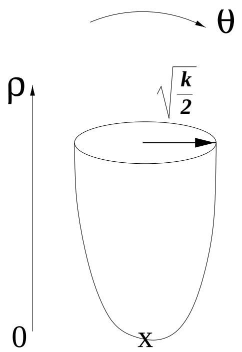

Figure 1: The 2d black hole.

where we have used the string frame and set $\alpha' = 1/2$. We can get some intuition into the spectrum of closed string modes from the minisuperspace approximation. To this end, we are looking for eigen-functions of the laplacian on the cigar,

$$\begin{aligned}\Delta &= -\frac{1}{e^{-2\Phi}\sqrt{\det G}}\;\partial_\mu e^{-2\Phi}\sqrt{\det G}\;G^{\mu\nu}\partial_\nu \\ &= -\frac{2}{k}\left[\partial_\rho^2 + (\coth\rho + \tanh\rho)\,\partial_\rho + \coth^2\rho\,\partial_\theta^2\right].\end{aligned} \tag{2.2}$$

The δ-function normalizable eigen-functions of this operator can be expressed in terms of hypergeometric functions through

$$\begin{aligned}\phi^j_{n0}(\rho,\theta) &= -\frac{\Gamma(-j+|n|/2)^2}{\Gamma(|n|+1)\Gamma(-2j-1)}\,e^{in\theta}\sinh^{|n|}\rho\;\times \\ &\quad\times F\left(j+1+\frac{|n|}{2}, -j+\frac{|n|}{2}, |n|+1; -\sinh^2\rho\right),\end{aligned} \tag{2.3}$$

where $j \in -1/2 + i\mathbb{R}$ describes the momentum along the ρ-direction of the cigar and $n \in \mathbb{Z}$ is the angular momentum under rotations around the tip. For the associated eigenvalues one finds

$$\Delta^j_{n0} = -\frac{2j(j+1)}{k} + \frac{n^2}{2k}\,. \tag{2.4}$$

In the symbols ϕ^j_{n0} and Δ^j_{n0} we have inserted an index '0' without any further comment. Our motivation will become clear once we start to look into the spectrum of the bulk CFT.

[2] A Born-Infeld analysis of branes in the $2D$ black hole has been worked out previously in [22].

[3] The solution for the D1-branes was already presented in [1] and [23].

Let us also stress that there are no L^2-normalizable eigenfunction of the laplacian on the cigar. Such eigenfunctions would correspond to *discrete states* living near the tip, but in the minisuperspace approximation one only finds exclusively *continuous states* which behave like plane waves at $\rho \to \infty$.

2.2 The spectrum of the bulk CFT

We now turn to the discussion of the conformal field theory (CFT) on the cigar which is known to possess a central charge $c = 2(k+1)/(k-2)$. Each of the wave functions (2.3) in the minisuperspace theory lifts to a *primary field* in this conformal field theory. But the full story must be a bit more complicated. In fact, at $\rho \to \infty$, the cigar looks like an infinite cylinder and for the latter we know that primary fields are labeled by momentum n and winding w around the compact circle, in addition to the continuous momentum $iP = j + \frac{1}{2}$ along the uncompactified direction. Hence, we expect that the full conformal field theory on the cigar has primary fields $\Phi^j_{nw}(z, \bar{z})$. Obviously, the winding modes $w \neq 0$ do not show up in the particle limit so that the minisuperspace analysis provides no help in deciding which values n, w and j can run through. The answer, however, is known and goes back to the work of Dijkgraaf, Verlinde and Verlinde [9]. It turns out that the closed string spectrum is made of two different series one of which is the *continuous series* with

$$j \in -\frac{1}{2} + i\,\mathbb{R}^+_0\,, \qquad n \in \mathbb{Z}\,, \qquad w \in \mathbb{Z}\,.$$

This is the series we have argued for above. But in addition there is also a *discrete series* of primary fields for which $w, n \in \mathbb{Z}$ are such that

$$j \,\in\, \mathcal{J}^d_{nw} := \left[\frac{1-k}{2}\,,\, -\frac{1}{2}\right] \cap \left(\mathbb{N} - \frac{1}{2}|kw| + \frac{1}{2}|n|\right). \tag{2.5}$$

This set of primary fields was first described in [9] with a slightly different lower bound on j. The bound we have written here appears in [10] (see also [24] for a related bound in the case of strings on $\mathrm{SL}(2,\mathbb{R})$). We shall confirm it later through world-sheet duality involving open strings. Let us also note that for $w = 0$, the discrete series is empty, in agreement with the minisuperspace analysis. The primary fields of these two series have conformal weights given by

$$h^j_{nw} = -\frac{j(j+1)}{k-2} + \frac{(n+kw)^2}{4k} \qquad \text{and} \qquad \bar{h}^j_{nw} = -\frac{j(j+1)}{k-2} + \frac{(n-kw)^2}{4k}\,. \tag{2.6}$$

Notice that these conformal weights are all positive, as for any euclidean unitary conformal field theory. In the limit of large level k, the sum $h^j_{n0} + \bar{h}^j_{n0}$ of the left and right conformal weights with $w = 0$ reproduces the spectrum (2.4) of the minisuperspace laplacian.

Over each of these primary fields there is a whole tower of descendants. The spectrum of conformal weights appearing in these towers is encoded in products of left- and right-moving chiral characters. For the continuous series, the chiral characters read as follows

$$\chi^c_{(j,\omega)}(q) = \mathrm{Tr}_{(j,\omega)}\, q^{L_0 - c/24} = \frac{q^{-\frac{(j+1/2)^2}{k-2} + \frac{\omega^2}{k}}}{\eta(q)^2}\,. \tag{2.7}$$

Here ω is some real number parametrizing the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ chiral algebra representations descending from a continuous representation of $\mathrm{SL}(2,\mathbb{R})$ of spin $j=-\frac{1}{2}+iP$. In the case of the discrete series, the expressions for chiral characters are a bit more complicated (see also appendix A),

$$\chi^d_{(j,\ell-j)}(q)=\frac{q^{-\frac{(j+\frac{1}{2})^2}{k-2}+\frac{(j-\ell)^2}{k}}}{\eta(q)^2}\left[\epsilon_\ell\sum_{s=0}^{\infty}(-1)^s q^{\frac{1}{2}s(s+2|\ell+\frac{1}{2}|)}-\frac{\epsilon_\ell-1}{2}\right] \tag{2.8}$$

where

$$\epsilon_\ell=\begin{cases}\ \ 1 \text{ if } \ell\geq 0\\ -1 \text{ if } \ell\leq -1\end{cases}.$$

Our formula for χ^d agrees with results in [25] as long as $\ell\geq 0$ (modulo the redefinition $j\to -j$).

Out of the characters we have introduced, one can build the full partition function of the bulk theory as follows

$$\begin{aligned}Z(q,\bar{q})\ =&\sum_{n,w\in\mathbb{Z}}\int dj\ N(j,n,w)\ \chi^c_{(j,(n+kw)/2)}(q)\,\chi^c_{(j,(-n+kw)/2)}(\bar{q})\,+\\&+\sum_{n,w\in\mathbb{Z}}\ \sum_{j\in\mathcal{J}^d_{nw}}\chi^d_{(j,(n+kw)/2)}(q)\,\chi^d_{(j,(-n+kw)/2)}(\bar{q})\,.\end{aligned} \tag{2.9}$$

Here, $N(j,n,w)$ is some nontrivial density of states for the continuous representations which can be found in [10]. Comparison of the conformal weights (2.6) with the chiral characters (2.7) and (2.8) shows that both in the discrete and in the continuous sectors, the closed string quantum numbers n,w are related to the left and right chiral representation labels $\omega,\bar{\omega}$ through

$$\omega=\frac{n+kw}{2}\,,\qquad \bar{\omega}=\frac{-n+kw}{2}\,. \tag{2.10}$$

Before we conclude this subsection let us remark that there exists also explicit formulas for the operator product expansions of the primary fields. We will not need them here in full generality and refer the interested reader to the literature (for instance, [11] and [26]–[28]). We will need, however, the following explicit expression for the two-point functions of primary fields

$$\left\langle\Phi^j_{nw}(z,\bar{z})\,\Phi^{j'}_{n'w'}(w,\bar{w})\right\rangle=\left[\delta(j+j'+1)+\mathcal{R}(j,n,w)\,\delta(j-j')\right]\ \frac{\delta_{n+n'}\,\delta_{w+w'}}{|z-w|^{2h^j_{nw}+2\bar{h}^j_{nw}}} \tag{2.11}$$

with

$$\mathcal{R}(j,n,w)=\nu_b^{2j+1}\ \frac{\Gamma(2j+1)}{\Gamma(-2j-1)}\ \frac{\Gamma(-j+\frac{n-kw}{2})\Gamma(-j+\frac{n+kw}{2})}{\Gamma(j+1+\frac{n-kw}{2})\Gamma(j+1+\frac{n+kw}{2})}\ \frac{\Gamma(1+b^2(2j+1))}{\Gamma(1-b^2(2j+1))} \tag{2.12}$$

and the standard notations

$$b^2=\frac{1}{k-2}\,,\qquad \nu_b=\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\,. \tag{2.13}$$

The quantity (2.12) is known as *reflection amplitude* since it describes the behavior of the fields Φ^j_{nw} under the reflection $j \to -j-1$,

$$\Phi^j_{nw}(z,\bar z) = \mathcal{R}(j,n,w)\Phi^{-j-1}_{nw}(z,\bar z)\,.$$

Observe that the bulk two-point functions become singular for fields from the discrete series because of the factors $\Gamma(-j+\frac{n\pm kw}{2})$. This singularity is caused by the specific normalization we have chosen in which the discrete fields are obtained from the continuous series through analytic continuation. Working with such a normalization is not really problematic since the divergent terms will cancel each other in all physical quantities. Let us also point out that due to the presence of the δ-function, the two-point functions are infinite whenever $j = j'$. The origin of this divergence is obvious: it is related to the infinite volume of the cigar and can be regularized by introducing a cut-off V_0. We will not do that explicitly, but eventually it is important to keep the issue in mind.

2.3 Coset construction from H_3

Our aim here is to outline how the quantities of the previous subsection can be recovered from the theory on the 3-dimensional euclidean space $H_3 \sim \mathrm{SL}(2,\mathbb{C})\ /\mathrm{SU}(2)$. The latter is related to $\mathrm{SL}(2,\mathbb{R})$ by a Wick rotation. An exact solution of this closed string background has been proposed in [29, 11] and it has subsequently been verified in [30]. Since the cigar geometry can be constructed as a coset $H_3\ /\mathbb{R}$ of H_3 the two conformal field theories are certainly closely related, though there exist some important subtleties.

Let us first understand how the descent from H_3 to the cigar works in the framework of the minisuperspace model. On H_3 we use a third coordinate $\tau \in \mathbb{R}$ in addition to the two coordinates ρ and θ we have used for the cigar (see figure 2). The metric on H_3 takes the form

$$ds^2 = \frac{k}{2}\,(d\rho^2 + \cosh^2\rho\, d\tau^2 + \sinh^2\rho\, d\theta^2)\,.$$

Furthermore, there is a non-trivial B-field and the dilaton on H_3 is constant. Using this information we can write down the laplacian Δ^{H} of the minisuperspace theory,

$$\Delta^{\mathrm{H}} \;=\; -\frac{2}{k}\,\left[\partial_\rho^2 + (\coth\rho + \tanh\rho)\,\partial_\rho + \cosh^{-2}\rho\,\partial_\tau^2 + \sinh^{-2}\rho\,\partial_\theta^2\right].$$

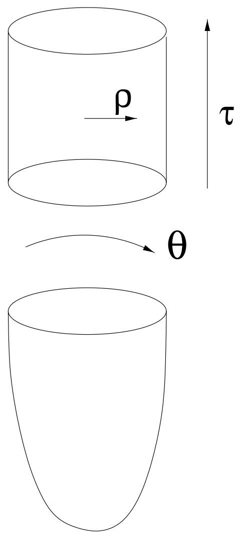

Figure 2: The cigar from H_3 .

Eigen-functions $\phi^{\mathrm{H},j}_{n,p}$ of this operator are labeled by three quantum numbers (j,n,p) where $p \in \mathbb{R}$ denotes the momentum along the τ-direction, i.e. the eigen-value of the operator $i\partial_\tau$. Explicit formulas for the eigen-functions of ϕ^{H} are easy to find (see e.g. [1]). From the following formula for the difference between the laplacians on H_3 and the cigar

$$\Delta^{\mathrm{H}} - \Delta = \frac{2}{k}\left[-\frac{1}{\cosh^2\rho}\,\partial_\tau^2 + \partial_\theta^2\right]$$

we can read off that eigen-functions and eigen-values of the laplacians in the two backgrounds are related by

$$\phi^j_n(\rho,\theta) = \phi^{\mathrm{H},j}_{n,p=0}(\rho,\theta,\tau)\,, \qquad \Delta^j_{n,0} = \Delta^{\mathrm{H},j}_{n,p=0} + \frac{n^2}{2k} = -\frac{2j(j+1)}{k} + \frac{n^2}{2k}\,. \tag{2.14}$$

Note that the formula for the eigen-values of the laplacian on the cigar agrees with the expression (2.4) we have given in the previous subsection. In passing let us remark that we can obtain a minisuperspace analogue of the reflection amplitude (2.12) from the corresponding quantity in H_3 (see e.g. [29]) through

$$\mathcal{R}_0(j,n) = \mathcal{R}^{\mathrm{H}}(j,n,p=0) = \frac{\Gamma(2j+1)\Gamma^2(-j+n/2)}{\Gamma(-2j-1)\Gamma^2(j+1-n/2)}\,.$$

This agrees with the results of the direct minisuperspace analysis in [9] and it coincides with the semi-classical limit of the exact result (2.12).

Let us now describe the coset construction for the full conformal field theory. As usual, the primary field of the coset $H_3/\mathrm{U}(1)$ can be constructed in the product theory $H_3 \times \mathrm{U}(1)$ of the H_3 conformal field theory with a 1-dimensional free space-like bosonic field. To be more precise, we continue by listing a few facts about the U(1)-model, i.e. the theory of a single free bosonic field X that is compactified on a circle of radius $R = \sqrt{2k}$. The field X possesses the propagator

$$\langle X(z,\bar z) X(w,\bar w)\rangle = \log|z-w|^2$$

and satisfies $X = X + 2\pi R$. The latter condition instructs us to consider the following set of local exponentials,

$$V_{n\,w}(z,\bar z) \;:=\; e^{i\frac{2\alpha}{\sqrt{2k}}X(z) + i\frac{2\bar\alpha}{\sqrt{2k}}\bar X(\bar z)}\,,$$

where $\alpha = \frac{1}{2}(n+kw)$ and $\bar\alpha = \frac{1}{2}(n-kw)$ and both n and w are integers. These exponential fields possess the following conformal dimensions

$$h_{n\,w} = \frac{(n+kw)^2}{4k} \qquad \text{and} \qquad \bar h_{n\,w} = \frac{(n-kw)^2}{4k}\,.$$

Let us also introduce the two chiral U(1) currents

$$J(z) = i\sqrt{\frac{k}{2}}\,\partial X(z,\bar z) \qquad \text{and} \qquad \bar J(\bar z) = i\sqrt{\frac{k}{2}}\,\bar\partial X(z,\bar z)\,.$$

With respect to these currents, our vertex operators $V_{n,w}$ carry the charge $(\alpha,\bar\alpha)$.

According to the usual rules of the coset construction, we can represent primary fields for the cigar as products of primary fields for the H_3-model and vertex operators in the U(1)-theory. Using the same conventions as in [1], we denote the primary fields of the H_3 model by $\Phi^{\mathrm{H,j}}_{n,p}$. These fields carry a charge $(\frac{1}{2}(n+ip), -\frac{1}{2}(n-ip))$ with respect to the H_3 currents $(J^0,\bar J^0)$ (which are some components of the $\mathrm{SL}(2,\mathbb{C})$ currents of the H_3 model). Hence, we can build primary fields $\phi^j_{n\,w}$ which are uncharged under the currents $J^0 + J$ and its conjugate $\bar J^0 - \bar J$ by (see [9]),

$$\Phi^j_{n\,w}(z,\bar z) = V_{n\,w}(z,\bar z)\,\Phi^{\mathrm{H,j}}_{n,-ikw}(z,\bar z)\,. \tag{2.15}$$

From this formula, the conformal dimensions of the coset field $\phi^j_{n\,w}$ can be easily determined, and the result agrees with eqs. (2.6).

To construct the characters of the coset model, we start from unspecialized characters of the $\mathrm{SL}(2,\mathbb{R})$ affine Lie algebra. For the continuous series of $\mathrm{SL}(2,\mathbb{R})$ representations in the H_3 model, the latter read

$$\chi^{\mathrm{H},c}_{(j,\beta)}(q,z) = \mathrm{Tr}_{(j,\beta)}\, q^{L_0-\frac{c}{24}}\, z^{J_0^0} = \frac{q^{-\frac{(j+\frac{1}{2})^2}{k-2}}}{\eta(q)^3} \sum_{r\in\mathbb{Z}} z^{r-\beta}\,,$$

where $j \in -\frac{1}{2}+i\mathbb{R}$, $\beta \in [0,1[$ and J_0^0 denotes the zero mode of the current J^0. We multiply these characters with the following unspecialized characters of the U(1) model,

$$\zeta_\alpha(q,z) = \frac{z^\alpha\, q^{\frac{\alpha^2}{k}}}{\eta(q)}\,, \tag{2.16}$$

to obtain characters of the product theory. Integration over the variable θ that appears in $z = \exp 2\pi i\theta$ along the entire real line gives the continuous characters $\chi^c_{(j,\alpha)}(q)$ of the coset model,

$$\int_{-\infty}^{\infty} d\theta\, \chi^{\mathrm{H,c}}_{j,\beta}(q,z)\zeta_\alpha(q,z) = \chi^c_{(j,\alpha)}(q)\,\eta(q)^{-2} \sum_{r\in\mathbb{Z}} \delta(\alpha-\beta+r)\,. \tag{2.17}$$

Similarly, we can descend from the unspecialized characters of the discrete $\mathrm{SL}(2,\mathbb{R})$ representations

$$\chi^{\mathrm{H},d}_j(q,z) = \frac{q^{-\frac{(j+\frac{1}{2})^2}{k-2}}\, z^{-j-\frac{1}{2}}}{i\vartheta_1(q,z)} \tag{2.18}$$

to the characters (2.8) of the discrete series in the cigar,

$$\int_{-\infty}^{\infty} d\theta\, \chi^{\mathrm{H,d}}_{j}(q,z)\zeta_\alpha(q,z) = \chi^d_{(j,\alpha)}(q)\,\eta(q)^{-2} \sum_{r\in\mathbb{Z}} \delta(j-\alpha+r)\,. \tag{2.19}$$

Actually, the integral over the real θ-axis is singular. In the following we shall adopt a regularization in which we shift the contour by an amount $i\epsilon$. The computation of the integral is a bit more involved in this case. It can be found in appendix A.

From the construction of the fields $\Phi^j_{n\,w}$ we can immediately determine their correlation functions. In particular, since the compactified free boson X does not contribute to the reflection amplitude, the cigar inherits its stringy reflection amplitude from H_3, i.e. $\mathcal{R}(j,n,w) = \mathcal{R}^{\mathrm{H}}(j,n,-ikw)$. An expression for $\mathcal{R}^{\mathrm{H}}$ can be found in [28].

3. The D0-branes

Our main aim is to discuss the exact solution for the point-like branes sitting at the tip of the cigar. We will start with a short semi-classical discussion in which we derive a minisuperspace formula for the coupling of closed strings to such branes. Then we present exact expressions for the boundary states and the open string spectra and we explain how they can be obtained by descent from H_3. Finally, we show that our formulas are consistent with world-sheet duality. In this computation we shall see that the contributions from the discrete series of closed string modes are crucial.

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3.1 Semi-classical description

It is rather obvious from a Born-Infeld analysis why point-like branes are expected to sit at the tip of the cigar. In fact, for a point-like brane, the Born-Infeld action reduces to

$$S_{BI} \propto \int d^d y\, e^{-\Phi}\, \sqrt{\det(G+B+F)} \propto \cosh\rho\,. \tag{3.1}$$

The string coupling takes its largest value at the tip so that the brane can minimize its mass at this point. Note that the semi-classical analysis does not provide any parameters for the D0-branes in the cigar. In the exact solution, on the other hand, we shall obtain a family of branes parametrized by one discrete parameter.

For comparison with the exact boundary state it is useful to spell out the semi-classical analogue of this quantity. In the minisuperspace model we only have wave-functions ϕ^j_{n0} corresponding to closed string modes with vanishing winding number $w=0$. Their coupling to the point-like brane at the tip of the cigar is simply given by the value of the function ϕ^j_{n0} at the point $\rho=0$. Using the explicit expression (2.3) for the wave-function that we provided in the previous section, we find

$$\left(\langle\Phi^j_{n0}\rangle^{\rm D0}\right)_{k\to\infty} = \phi^j_{n0}(\rho=0) = -\frac{\Gamma(-j)^2}{\Gamma(-2j-1)}\delta_{n=0}\,. \tag{3.2}$$

Note that due to the rotational symmetry of the point-like brane, only the modes with angular momentum $n=0$ have a non-vanishing coupling. This will remain true for the exact answer, but the coupling of the mode with $n=0$ will acquire a non-trivial dependence on the parameter $b=(k-2)^{-1/2}$ that controls the radius of the circle at infinity.

We can also predict some features of the open string spectrum of the D0-brane. Since this brane is point-like, the spectrum should be discrete. Moreover, the rotation symmetry of the D0-brane allows the presence of states with arbitrary momentum n in the open string spectrum, and forbids the presence of winding open strings. These expectations will be confirmed in the sequel.

3.2 The exact solution

Let us now summarize our results for the D0-branes in the cigar and compare them to semi-classical expectations. We claim that there exists a discrete family of localized branes living near the tip of the cigar, parametrized by some integer $m=1,2,\ldots$.[4] For the one-point functions of the bulk primary fields in the presence of these branes we propose

$$\begin{aligned}\langle\Phi^j_{nw}(z,\bar z)\rangle^{\rm D0}_m = \delta_{n,0}\,\mathcal{N}_m(b)\,(-1)^{mw}\left(\frac{k}{2}\right)^{1/4}\frac{\Gamma(-j+kw/2)\Gamma(-j-kw/2)}{\Gamma(-2j-1)}\times \\ \times\frac{\sin\pi b^2}{\sin\pi b^2 m}\,\frac{\sin\pi b^2 m(2j+1)}{\sin\pi b^2(2j+1)}\,\frac{\Gamma(1+b^2)\nu_b^{j+1}}{\Gamma(1-b^2(2j+1))}\,\frac{1}{|z-\bar z|^{h^j_{nw}+\bar h^j_{nw}}}\,.\end{aligned}$$

[4] Within this section we allow m to be any integer. There are some indications, however, that $m=1$ is the only value of m that corresponds to a physical brane in the $2D$ black hole (see below).

In principle, there is some freedom in normalizing the one-point functions which gives rise to the numerical pre-factor $\mathcal{N}_m(b) = \langle \Phi^{-1}_{00}(z = \frac{i}{2}) \rangle$. Here, we fix the value of this quantity to be

$$\mathcal{N}_m(b) = \left(\frac{b^2}{2}\right)^{1/4} \frac{\sin \pi b^2 m}{\sqrt{2\pi} \sin \pi b^2} \,. \tag{3.3}$$

This choice is distinguished through our Cardy computation (see below) and it also arises when we descend from properly normalized boundary states in the H_3 model (see appendix B and next subsection).

There are several remarks we would like to make about our formula for the one-point function. To begin with, we should stress that it does not only give the couplings to states from the continuous series but that the expression is also meant to apply to the discrete spectrum of the theory. For the latter, however, the quantity on the right hand side is singular. We anticipated such singularities before in our discussion of the reflection amplitude (2.12) for bulk fields. They were explained there as an artifact of our normalization. If we would renormalize fields from the discrete series such that they have a finite two-point function, then their one-point functions would become finite as well.

It is also worth pointing out that our formula shares many features with the related expression for point-like branes in Liouville theory [17]. In our case, however, there is only a single discrete parameter m instead of two, and all the associated branes possess a semi-classical limit $b \to 0$. If we set $m = 1$ and restrict to states with zero winding number w, we recover the semi-classical prediction (3.2). Brane couplings for $m \neq 1$ contain an additional factor m when b goes to zero. Hence, in the limiting regime they look like a collection of m point-like branes at the tip of the cigar.

The second result we want to state here concerns an expression for the spectrum of open strings stretching between two D0-branes with parameters m and m', respectively. We express this in the form of an annulus amplitude with modular parameter $q = \exp 2\pi i\tau$,

$$Z^{\mathrm{D0}}_{mm'}(q) = \sum_{2J+1=\max(m,m')}^{m+m'-1} \sum_{\ell \in \mathbb{Z}} \left[\chi^d_{(J,\ell-J)}(q) - \chi^d_{(-J-1,\ell+J+1)}(q) \right] . \tag{3.4}$$

The representation labels $\ell - J$ and $\ell + J + 1$ should be interpreted as the momentum of the open strings in the angular direction, whereas their winding is zero. The latter fact can be traced back to the rotational symmetry of the point-like branes, which prevents winding open strings from living on them (and also implies that closed strings coupling to them must have zero angular momentum). Of course, even though discrete states with zero winding do not appear in the closed string spectrum, this does not prevent them from being physical in the open string theory. A potentially more severe problem is the appearance of larges values of J for m large enough, since it threatens the unitarity of the spectrum. We take this as one indication that the large values of the parameter m are unphysical.

For two equal branes with the special labels $m = 1 = m'$, our formula (3.4) for the partition function of D0-branes simplifies as follows,

$$Z^{\mathrm{D0}}_{11}(q) = \sum_{\ell \in \mathbb{Z}} \left[\chi^d_{(0,\ell)}(q) - \chi^d_{(-1,\ell-1)}(q) \right] = q^{-\frac{c}{24}} \left(1 + 2q^{1+\frac{1}{k}} + q^2 + O(q^{2+\frac{1}{k}}) \right) . \tag{3.5}$$

As for the point-like brane in Liouville theory, there appear no contributions from states with conformal weight $h = 1$. This means that the point-like branes do not possess marginal deformations that can move them away from the tip of the cigar, in agreement with our geometric intuition. However, in the limit of large k, two states become marginal. They correspond to displacements of a D0-brane in the flat space limit of the 2-dimensional cigar.

Note also that at finite k the identity field is the only relevant operator of the theory. The same is true for a stack of such branes and, unlike in many other backgrounds, their spectrum does not even contain marginally relevant fields. In a supersymmetric theory, the identity field is projected out and hence a stack of N D0-branes with parameter $m = 1$ appears to be a stable state with D0-brane charge N. But our theory contains a candidate for another state with the same charge, namely the D0-brane with parameter $m = N$. At least in the semi-classical limit, the two D0-brane charges indeed agree. On the other hand, it is not hard to see that the D0-brane with parameter $m = N$ has lower mass than the stack of D0-branes with parameter $m = 1$. Recall that the mass of a D-brane is the one-point function of the identity operator. Hence, up to some k-dependent pre-factor, the mass of the N^{th} D0-brane is given by $\sin \pi b^2 N$ which is strictly smaller than the total mass $N \sin \pi b^2$ of the stack. We can only reconcile this with the stability of the stack if we declare the D0-branes with $m > 1$ to be unphysical. Support for this solutions also comes from the analysis of the spectrum of open strings that stretch between D0 and D2-branes (see below). Let us finally also point out that even in a supersymmetric theory the D0-brane spectra with $m > 1$ contain a large number of tachyonic modes and hence would be unstable.

3.3 D0-branes from descent

We have to justify our expressions for the boundary states (3.3) and spectrum (3.4) of the D0-branes in the cigar. Our basic strategy is to "descend" from the known S^2 -branes in H_3 [1]. The full consistency of those S^2 -branes has not been proved, but they satisfy two nontrivial consistency checks. First, their boundary state was shown to solve constraints coming from the factorization of bulk two-point functions [1, 31]. Furthermore, consistency with world-sheet duality has been checked. Since these branes are compact, the annulus amplitude is built from a discrete set of open string modes. A generalization of this Cardy-type computation is spelled out in appendix B.

Let us point out that the S^2 -branes in H_3 do not have a clear geometrical interpretation. In fact, they look like spheres with an *imaginary* radius (see [1]). This is not too surprising since H_3 also suffers from an imaginary Neveu-Schwarz B-field. On the other hand, such pathologies are cured when we pass down to the cigar and consequently the D0-branes are good, physical objects.

To obtain the exact expression for the coupling of closed string modes to the D0-branes we follow the simple prescription that is encoded in the product formula (2.15) for the bulk field Φ^j_{nw}. In other words, we construct the one-point functions for the cigar as a product of one-point functions for an S^2 -brane in H_3 with a one-point function of the U(1) vertex operator V_{nw}. Following the usual ideology of brane constructions in coset models [32, 33, 34], we have to impose Neumann boundary conditions on the free boson.

In fact, this choice guarantees that the resulting branes in the numerator theory extend in one direction, along the orbits of the U(1) action. After passing to the coset, we are left with a compact object again.

It is now easy to explain the origin of the formula (3.3). To this end we rewrite the one-point function of the S^2 brane in H_3 (see [1]) in terms of the continuous momentum p and the angular momentum n,

$$\langle \Phi^{\mathrm{H},j}_{np}(z,\bar z)\rangle^{S^2}_m \;=\; \delta_{n,0}\,\mathcal{N}_m(b)\,\frac{\Gamma(-j+ip/2)\Gamma(-j-ip/2)}{\Gamma(-2j-1)}\times \qquad (3.6)$$

$$\times\,\frac{\sin\pi b^2}{\sin\pi b^2 m}\,\frac{\sin\pi b^2 m(2j+1)}{\sin\pi b^2(2j+1)}\,\frac{\Gamma(1+b^2)\nu_b^{j+1}}{\Gamma(1-b^2(2j+1))}\,\frac{1}{|z-\bar z|^{-\frac{2j(j+1)}{k}}}\,.$$

Here, $\mathcal{N}_m(b)$ is the same factor (3.3) as in the last subsection. Though this factor has not been spelled out in [1], it is implicit in the discussion and we compute it explicitly in appendix B. As explained above, the expression (3.6) needs to be multiplied with the following one-point function for the U(1) vertex operators

$$\langle V_{nw}(z,\bar z)\rangle^N = \delta_{n,0}\,\frac{(k/2)^{1/4}}{|z-\bar z|^{\frac{kw^2}{2}}}\,, \qquad (3.7)$$

where the superscript N stands for Neumann boundary conditions. If we finally replace p by $-ikw$, we end up with the expression (3.3) for the one-point function in the presence of D0-branes.[5]

In order to deduce our formula (3.4) for the open string spectrum on the D0-branes, we start from the unspecialized partition function (B.4) of the S^2 -branes. Each character in this partition function can be decomposed into Fourier modes labelled by an integer ℓ. This corresponds to the decomposition of affine $\mathrm{SL}(2,\mathbb{R})$ representations into sectors of the non-compact parafermions. Our prescription for the open-string spectrum on the D0-branes is simply to sum all those coset representations, just as in the construction of B-branes in SU(2) (2)/ U(1) from maximally symmetric branes in SU(2) (2) [32].

3.4 Cardy consistency condition

We are now prepared to demonstrate that our exact solution is consistent with world-sheet duality, i.e. to show that after modular transformation the annulus amplitude (3.4) can be re-interpreted as an overlap of the boundary states which are encoded in the one-point functions (3.3). Though this computation can in principle be performed without any input from the H_3 theory, using some of the quantities associated with the spherical branes on H_3 turns out to be quite useful. The coset model computations, however, are more complicated because they involve contributions from discrete closed string states. There are no such discrete states in the bulk spectrum of H_3 and hence the overlap of the boundary states for spherical branes in H_3 contains no discrete contributions.On the cigar such terms are claimed to appear and we will see that this is indeed the case.

[5]The extra factor $(-1)^{mw}$ in eq.$\tilde{}$(3.3) has been introduced to ensure consistency with the world-sheet duality (see below).

Our strategy here is to start on the open string side, i.e. with the amplitude (3.4). The modular transformation to the closed string world-sheet is then performed in two steps. First we rewrite the open string amplitude on the cigar in terms of the corresponding quantity for spherical branes on H_3 using the following formula,

$$Z^{\rm D0}_{mm'}(q) = \eta^2(q) \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\alpha\, \zeta_\alpha(q,z) Z^{S^2}_{mm'}(q,z)\,. \tag{3.8}$$

It follows directly from our description of the coset construction (see formula (2.19)) and can be verified by inserting the explicit expressions (3.4) and (B.4) for the two annulus amplitudes. In a second step, we modular transform the annulus amplitude of the spherical brane. Some relevant formulas can be found in appendix B (see in particular eq.(B.2)). After having rewritten the partition function in this way, we arrive at the expression

$$Z^{\rm D0}_{mm'}(q) = -2\eta(q)\sqrt{\frac{b^2k}{-i\tau}} \int d\theta \int dP\, \frac{\tilde{q}^{b^2P^2}}{\vartheta_1(\tilde{q},\tilde{z})} \sum_J \sinh 2\pi b^2 P(2J+1)\, \sinh 2\pi P\tilde{\theta}\,. \tag{3.9}$$

Here, $q = \exp 2\pi i\tau$ and the summation over J has the same range as in eq. (3.4) so that, in particular, $(2J+1)$ is an integer. We also recall that the parameter $\tilde{\theta}$ appearing in $\tilde{z} = \exp 2\pi i\tilde{\theta}$ is given by $\tilde{\theta} = \theta/\tau$. In the derivation of eq. (3.9) from eq. (3.8) we have performed the integral over σ.

Now we want to invert the integrations over θ and P. But the integrand has poles along the real θ-axis. Hence, we first have to displace the θ integration contour slightly, away from the real axis to $\mathbb{R}+i\epsilon$. For definiteness we will assume that $\epsilon > 0$, but the final answer does not depend on the sign of ϵ. Using elementary techniques from the theory of complex functions it is possible to prove the following formula

$$\int_{\mathbb{R}+i\epsilon} d\theta\, \frac{\sinh 2\pi P\tilde{\theta}}{\vartheta_1(\tilde{q},\tilde{z})} = \frac{\tanh \pi P}{\sqrt{-i\tau}\,\eta(q)^3} \sum_{n\in\mathbb{Z}} (-1)^n\, \tilde{q}^{n^2/2 - iPn}\,. \tag{3.10}$$

Insertion into our previous expression (3.9) for the amplitude $Z^{\rm D0}_{mm'}(q)$ gives

$$Z^{\rm D0}_{mm'}(q) = -2\sqrt{b^2k} \int dP\, \sum_J \sinh 2\pi b^2 P(2J+1)\, \frac{\tanh\pi P}{\eta(\tilde{q})^2} \sum_{n\in\mathbb{Z}} (-1)^n\, \tilde{q}^{b^2(P-\frac{i}{2b^2}n)^2 + \frac{k}{4}n^2}\,. \tag{3.11}$$

On the right hand side we have some series containing powers of the parameter $\tilde{q}$, but the exponents are complex. This prevents a direct interpretation of these exponents as energies of closed string states which would couple to the D-branes. To cure the problem, we exchange the summation of n with the integration over P and we substitute P by the new variable

$$P_n = P - \frac{i}{2b^2}\, n$$

in each of the summands. P_n is integrated along the line $\mathbb{R} - in/2b^2$. The crucial idea now is to shift all these different integration contours back to the real line. This will give contributions associated with the continuous part of the boundary state. But while we shift the contours, we pick up residues from the singularities. The latter lead to terms associated with the discrete series. To work out the details, we split the partition function into a continuous and a discrete piece,

$$Z^{\rm D0}_{mm'}(q) = Z^c_{mm'}(q) + Z^d_{mm'}(q)\,.$$

Note that this split is defined with respect to the closed string modes. In terms of open string modes, our partition function contains only discrete contributions. According to our description above, the continuous part of the partition function reads as follows

$$Z^c_{mm'}(q) = -2\sqrt{b^2k}\int dP \sum_{w\in\mathbb{Z}} \frac{\tilde{q}^{b^2P^2+\frac{k}{4}w^2}}{\eta(\tilde{q})^2} \sum_J (-1)^{2Jw}\, \frac{\sinh 2\pi b^2 P(2J+1)\sinh 2\pi P}{\cosh 2\pi P + \cos \pi k w}\,. \tag{3.12}$$

In writing this formula we have renamed the summation index from n to w. It is now not difficult to see that this part of our partition function can be expressed as follows

$$Z^c_{mm'}(q) = \int dP \sum_{w\in\mathbb{Z}} \chi^c_{(-\frac{1}{2}+iP,\frac{kw}{2})}(\tilde{q})\, \Psi_m\left(-\frac{1}{2}+iP,w\right)\, \Psi_{m'}\left(-\frac{1}{2}+iP,w\right)^*$$

where

$$\begin{aligned}\Psi_m(j,w) \,&=\, (-1)^{mw}\left(\frac{kb^2}{\pi^2}\right)^{1/4} \frac{\Gamma(-j+kw/2)\Gamma(-j-kw/2)}{\Gamma(-2j-1)}\,\times \\ &\times \frac{\sin\pi b^2 m(2j+1)}{\sin\pi b^2(2j+1)}\, \frac{\Gamma(1+b^2)\nu_b^{j+1}\sqrt{\sin\pi b^2}}{2\Gamma(1-b^2(2j+1))}\,.\end{aligned} \tag{3.13}$$

The coefficients $\Psi_m(-\frac{1}{2}+iP,w)$ of the boundary state that we have introduced to present our result for Z^d are easily recognized as the couplings of the one-point functions (3.3).

After this first success we now turn to the calculation of the discrete piece of the amplitude. It is clear from (3.11) that the residues we pick up while shifting the contours give the following discrete contribution to the partition function

$$\begin{aligned}\frac{1}{2}Z^d_{mm'}(q) \,&=\, -2\sqrt{b^2k}\sum_J\sum_{n=1}^{\infty}(-1)^n\tilde{q}^{\frac{k}{4}n^2}\,\times \\ &\times \sum_{m=0}^{E(\frac{k-2}{2}n-\frac{1}{2})} \sin\left(\frac{2\pi(2J+1)(m+1/2)}{k-2}\right)\, \frac{\tilde{q}^{-\frac{1}{k-2}\left(m+\frac{1}{2}-\frac{k-2}{2}n\right)^2}}{\eta(\tilde{q})^2}\,.\end{aligned} \tag{3.14}$$

Here, $E(x)$ denotes the integer part of x. A careful study of the energies which appear in this partition sum shows that the contributing states can be mapped to discrete closed string states with zero momentum. The latter are parametrized by their winding number w and by their spin j or, equivalently, by the label $\ell = \frac{kw}{2} + j \in \mathbb{Z}$, and the level number s as in the character (2.8). The map between the parameters (w, ℓ, s) and the summation indices (m, n) of formula (3.14) is

$$m = \ell - w\,, \qquad n = s + w\,. \tag{3.15}$$

It can easily be inverted to compute the labels (w, ℓ, s) in terms of (m, n),

$$w = E\left(\frac{2m+1}{k-2}\right), \qquad \ell = m + E\left(\frac{2m+1}{k-2}\right), \qquad s = n - E\left(\frac{2m+1}{k-2}\right). \tag{3.16}$$

In terms of j and w, the partition function (3.14) may now be rewritten as follows,

$$Z^d_{mm'}(q) = -2\sqrt{b^2k}\sum_J\sum_{w\in\mathbb{Z}}\sum_{j\in\mathcal{J}^d_{0w}}(-1)^{2Jw}\sin\left(\frac{\pi(2J+1)(2j+1)}{k-2}\right)\chi_{(j,\frac{kw}{2})}(\tilde q). \quad (3.17)$$

It remains to verify that the coefficients of the characters coincide with those derived from the boundary state (3.3). In our normalization, the boundary coefficients $\Psi_m(j,w)$ are given through the same expression (3.13) as for the continuous series but we have to divide each term in the annulus amplitude by the non-trivial value of the bulk two-point function (2.11) of discrete closed string states, i.e.

$$\mathcal{Z}^d_{mm'}(q) = \sum_{w\in\mathbb{Z}}\sum_{j\in\mathcal{J}^d_{0w}}\frac{\Psi_m(j,w)\Psi_{m'}(j,w)^*}{\left\langle\Phi^j_{0w}\Phi^j_{0w}\right\rangle}\chi^d_{(j,\frac{kw}{2})}(\tilde q) \quad (3.18)$$

$$= \sum_{w\in\mathbb{Z}}\sum_{j\in\mathcal{J}^d_{0w}}\mathrm{Res}_{x=j}\left(\frac{\Psi_m(x,w)\Psi_{m'}(x,w)^*}{\mathcal{R}(x,n=0,w)}\right)\chi^d_{(x,\frac{kw}{2})}(\tilde q)\,. \quad (3.19)$$

The second line provides a more precise version of what we mean in the first line. Recall that the bulk two-point correlator contains a δ-function which arises because of the infinite volume divergence. If we drop this δ-function in the denominator, then the result has poles and the physical quantities are to read off from the residues. A short explicit computation shows that the argument of the Res-operation in eq. (3.19) indeed has simple poles at $x\in\mathcal{J}^d_{0w}$ and that the residues agree exactly with the coefficients in formula (3.17), just as it is required by world-sheet duality.

4. The D1-branes

We now turn the construction of a 2-parameter family of 1-dimensional branes on the cigar. All these branes stretch out to $\rho=\infty$ and hence they are non-compact. Our discussion follows the pattern of the previous section. We shall begin with some remarks on the semi-classics and then spell out the one-point function and the open string spectrum. Because of the non-compactness of the branes' world-volumes, their open string spectrum is continuous and it involves some non-trivial spectral density. Both quantities, the exact one-point functions and the open string spectra, are obtained rather easily from the corresponding quantities for the euclidean AdS_2 branes in H_3 (see [1, 18]) since there are no complications associated with the discrete series on the cigar. The section concludes with a Cardy-type consistency check of world-sheet duality.

4.1 Semi-classical description

The D1-branes in the cigar are most easily studied using a new coordinate $u=\sinh\rho$ along with the usual angle θ. We have $u\geq 0$ and $u=0$ corresponds to the point at the tip of the cigar. In the new coordinate system, the background fields read

$$ds^2 = \frac{k}{2}\,\frac{du^2+u^2d\theta^2}{1+u^2}\,,\qquad e^\Phi = \frac{e^{\Phi_0}}{(1+u^2)^{1/2}}\,. \quad (4.1)$$

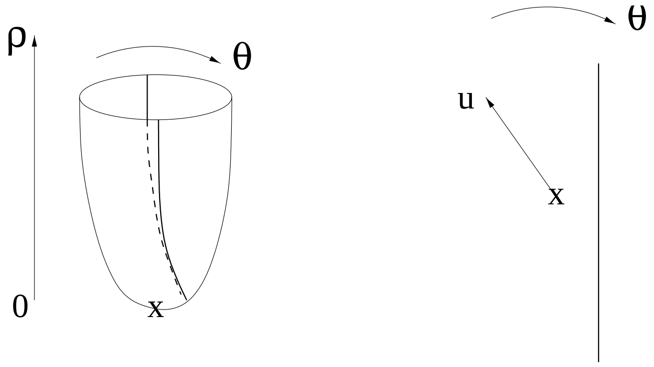

Figure 3: A D1-brane in the cigar in ρ and u coordinates.

When we insert these background data into the Born-Infeld action for 1-dimensional branes we obtain

$$S_{\mathrm{BI}} \propto \int dy\, \sqrt{u'^2 + u^2\theta'^2}\,, \tag{4.2}$$

where we used the same conventions as in section 3.1 and the primes denote derivatives with respect to the world-volume coordinate y on the D1-brane. It is now easy to read off that D1-branes are straight lines in the plane $(u = \sinh\rho, \theta)$. These are parametrized by two quantities, one being their slope, the other the transverse distance from the origin. In our original coordinates ρ, θ, this 2-parameter family of 1-dimensional branes is characterized by the equations

$$\sinh\rho\ \sin(\theta - \theta_0) = \sinh r\,. \tag{4.3}$$

Note that the brane passes through the tip if we fix the parameter r to $r = 0$. All branes reach the circle at infinity ($\rho = \infty$) at two opposite points. The positions θ_0 and $\theta_0 + \pi$ of the latter depend on the second parameter θ_0 (see figure 3).

Now that we have some idea about the surfaces along which our branes are localized we can calculate their coupling to closed string modes in the semi-classical limit. This is done by integrating the minisuperspace wave functions (2.3) of closed string modes over the 1-dimensional surfaces (4.3). The result of this straightforward computations is the prediction

$$\Big(\langle \Phi^j_{n0}\rangle^{\mathrm{D1}}_r\Big)_{k\to\infty} = e^{in\theta_0}\,\frac{\Gamma(2j+1)}{\Gamma(1+j+n/2)\Gamma(1+j-n/2)}\,\Big(e^{-r(2j+1)} + (-1)^n e^{r(2j+1)}\Big)\,. \tag{4.4}$$

The minisuperspace model of the cigar does not include any states associated with closed string modes of non-vanishing winding number. But in the case of the D1-branes, experience from the analysis of branes on a 1-dimensional infinite cylinder teaches us that closed string modes with $w \neq 0$ do not couple at all. Since the discrete closed string modes only appear at $w \neq 0$, they are like-wise not expected to couple to the D1-branes. Consequently, our formula (4.4) predicts the semi-classical limit of the only non-vanishing couplings in the theory.

Let us now come to the semi-classical analysis of the open string spectrum. Because of the non-compactness of the 1-dimensional branes, this is a much more interesting topic than for the D0-branes. From the geometry we have just described, it is rather obvious

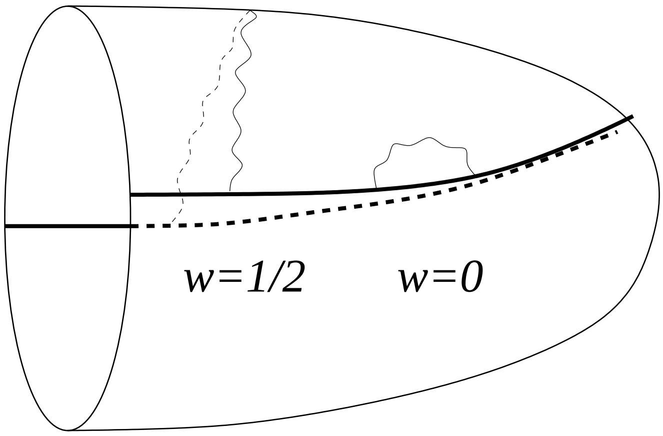

Figure 4: A D1-brane in the cigar with open strings on it.

that the spectrum of open strings on the D1-branes cannot contain any momentum modes along the θ direction. Open string winding modes around the compact circle, however, will appear in the exact spectrum and since there are two branches of our branes at large ρ, the winding number is expected to be a half-integer number (see figure 4). Note that this winding number is certainly not conserved in physical processes since open strings can 'unwind' in the interior where the two branches of our 1-dimensional branes come together.

It is also clear from the geometry of the D1-branes that open strings can be sent in from infinity with any non-negative real momentum P along the radial direction. In other words, open strings on the 1-dimensional branes possess a continuous spectrum. The latter can be characterized by an interesting new quantity: an open string spectral density. As we have argued before, the minisuperspace model cannot say anything on strings with non-zero winding number. Open strings with zero winding, however, possess a semi-classical point-particle limit. Hence, for the modes with $w = 0$, we can make some prediction for the density $N(j, w = 0|r, r)$ of radial open string momenta $P = -i(2j+1)$. In fact, by very general arguments (see e.g. [1]), the quantity ρ is related the so-called reflection amplitude $R(j)$ by[6]

$$N\left(-\frac{1}{2} + iP\right) \sim \frac{1}{2\pi i}\frac{\partial}{\partial P} R\left(-\frac{1}{2} + iP\right). \tag{4.5}$$

By definition, the reflection amplitude is the ratio between the coefficients of the in-coming an out-going plane waves within a wave function. For the minisuperspace limit of open strings on D1 branes we can compute $R(j)$ either through a direct study of the 1-dimensional laplacian on the D1-branes or by descent from the corresponding minisuperspace model of open string on euclidean AdS_2 -branes. Both ways lead to the same answer,

$$R(P, w = 0|r, r)_{k\to\infty} = -(\cosh r)^{2iP}\frac{\Gamma(1 - iP)\Gamma(\frac{1}{2} + iP)}{\Gamma(1 + iP)\Gamma(\frac{1}{2} - iP)}. \tag{4.6}$$

[6]A more careful discussion must take into account that the spectral density itself is divergent. The divergence, however, is universal, and it can be removed, e.g. by considering relative spectral densities.

Because of the rotational symmetry, the reflection amplitude does not depend on the angular parameter θ_0. The geometrical picture we have outlined in this subsection along with the two precise predictions (4.4)–(4.6) we made for the semi-classical limit of the bulk one-point functions and the open string spectral density will be confirmed nicely through our exact solution.

4.2 The exact solution

In this subsection we shall present our proposal for exact boundary conformal field theory quantities describing 1-dimensional branes on the cigar. We claim that the exact solution is parametrized by two continuous parameters r and θ_0, just as in the semi-classical limit. Expressions for the one-point functions of closed string fields Φ in the D1-brane background were already proposed in [1],

$$\begin{aligned}\langle \Phi^j_{nw}(z,\bar{z})\rangle^{\mathrm{D1}}_{(r,\theta_0)} \;=\; & \delta_{w,0}\,\mathcal{N}'(b)\, e^{in\theta_0}\,(2k)^{-\frac{1}{4}}\,\frac{\Gamma(2j+1)}{\Gamma(1+j+n/2)\Gamma(1+j-n/2)}\;\times \\ & \times\left(e^{-r(2j+1)}+(-1)^n e^{r(2j+1)}\right)\Gamma(1+b^2(2j+1))\nu_b^{j+1/2}\frac{1}{|z-\bar{z}|^{h^j_{nw}+\bar{h}^j_{nw}}}\,,\end{aligned} \tag{4.7}$$

where the prefactor is $\mathcal{N}'(b) = (8b^2)^{-1/4}$. Of course, the expression (4.7) correctly reproduces the semi-classical expression (4.4). The precise matching justifies the identification of the parameter r and θ_0 in the one-point function with the geometrical parameters r and θ_0 from the previous subsection. We also point out that the one-point functions possess poles whenever $b^2(2j+1) = ib^2P$ is a negative integer. As in the case of extended branes in Liouville theory [16], these poles can be traced back to the non-compactness of our branes.

In addition, we want to spell out the density of open strings living on a D1-brane. For symmetry reasons this depends only on the parameter r, not on the angle θ_0. As usual, the large volume divergence of the spectral density can be regularized by introducing a cut-off in the coordinate ρ. Since the divergence of the spectral density is universal, the cut-off can be sent to infinity in relative spectral densities. To be more specific, we will subtract the spectral density of the brane with label $r = 0$, i.e. consider the quantity

$$\Delta N(P,w|r,r) := \frac{1}{2\pi i}\frac{\partial}{\partial P}\log\frac{R(-1/2+iP,w|r,r)}{R(-1/2+iP,w|0,0)}\,. \tag{4.8}$$

In this formula we have expressed the relative spectral density through a quotient of the associated stringy reflection amplitude. We can now express our proposal for the relative spectral density in terms of formulas for $R(-\frac{1}{2}+iP,w|r,r)$. As we observed previously, the open string winding number w is expected to take values in $w \in \frac{1}{2}\mathbb{Z}$. Our formulas for $R(j,w)$ will turn out to depend on whether w is integer or not. In the former case, $R(j,w|r,r)$ is given by

$$R(-\frac{1}{2}+iP, w\in\mathbb{Z}|r,r) = \nu_b^{iP}\,\frac{\Gamma_k^2(\frac{1}{2}-iP+\frac{1}{b^2})}{\Gamma_k^2(\frac{1}{2}+iP+\frac{1}{b^2})}\,\frac{\Gamma_k(2iP+\frac{1}{b^2})}{\Gamma_k(-2iP+\frac{1}{b^2})}\,\frac{S_k^{(i)}(\frac{r}{\pi b^2}+P)}{S_k^{(i)}(\frac{r}{\pi b^2}-P)} \tag{4.9}$$

with

$$\log S_k^{(i)}(x) = \log S_k^{(0)}(x) = i\int_0^\infty \frac{dt}{t}\left(\frac{\sin 2tb^2x}{2\sinh b^2t\sinh t}-\frac{x}{t}\right). \tag{4.10}$$

Note that the reflection amplitude does not depend on the open string winding number w as long as it is integer. For reasons that we will review below, open strings with integer winding number on a D1-brane in the cigar possess the same spectral density as open strings on an AdS_2 brane in AdS_3 [1]. Similarly, for open string modes with $w \in \frac{1}{2} + \mathbb{Z}$ we propose a reflection amplitude of the form (4.9) but with

$$\log S_k^{(i)}(x) = \log S_k^{(1)}(x) = i \int_0^\infty \frac{dt}{t} \left(\frac{\cosh t \sin 2tb^2 x}{2 \sinh b^2 t \sinh t} - \frac{x}{t} \right) . \tag{4.11}$$

The associated spectral density coincides with the spectral density of open strings stretched between two opposite AdS_2 branes in AdS_3 [18]. For this reason we shall often denote $\Delta N(P,w), w \in \frac{1}{2} + \mathbb{Z}$, by $\Delta N(P|r,-r)$, and we shall also use the symbol $R(P|r,-r)$ for the reflection amplitude $R(P,w|r,r)$ with $w \in \frac{1}{2} + \mathbb{Z}$.

Finally, we provide a prescription for how to compute the density of open strings stretching between two D1-branes with different labels r, r'. At the same time we want both branes to possess identical angular parameter θ_0. This implies that the two branes coincide at $\rho = \infty$ so that the winding number w continues to take values in $\frac{1}{2}\mathbb{Z}$. The density of open strings stretching between two such branes (r, θ_0) and (r', θ_0) is given by (see [18] for a related formula in the case of AdS_2 branes in AdS_3)

$$\Delta N(P,w|r,r') = \Delta N \left(P, w \left| \frac{r+r'}{2}, \frac{r+r'}{2} \right. \right) + \Delta N \left(P, w \left| \frac{r-r'}{2}, \frac{r'-r}{2} \right. \right) . \tag{4.12}$$

This relative spectral density features in the following expression for the relative open string partition function of the D1-branes,

$$\left(Z^{\rm D1}_{(r,\theta_0)(r',\theta_0)} - Z^{\rm D1}_{(0,\theta_0)(0,\theta_0)} \right)(q) = \int dP \sum_{w \in \frac{1}{2}\mathbb{Z}} \Delta N(P,w|r,r') \, \chi^c_{(-\frac{1}{2}+iP,w)}(q) \quad . \tag{4.13}$$

In the special case $r = r'$ we shall later reproduce quantity (4.13) through world-sheet duality starting from the one-point functions for D1-branes that we have proposed above.

Let us also briefly point out that for $r = r'$ and $w = 0$, the semi-classical limit of our expression for the relative spectral density ΔN agrees with the expression (4.6) above. In fact, with the choices we have made it is easy to show that

$$\lim_{k \to \infty} \Delta N(P,0|r,r) = \frac{1}{\pi} \int_0^\infty \frac{dt}{t} \left(\frac{\cos \frac{2tr}{\pi}}{\sinh t} - \frac{1}{t} \right) = \frac{1}{\pi} \log \cosh r \, . \tag{4.14}$$

We would like to stress that the quantity $\Delta N(r,-r)$ which appears in $\Delta N(r,r')$ does not possess a well defined semi-classical limit. Given the geometrical setup we have drawn above, this does not come as a big surprise, but it still can be considered a test of the conformal field theory solution.

For completeness, let us mention that we also expect discrete open string states to live on D1-branes in the cigar. This is a consequence of similar results for AdS_2 branes in AdS_3 (reviewed in [23]). However, this discrete part of the spectrum does not depend on the brane's parameter and thus does not contribute to relative spectral densities.

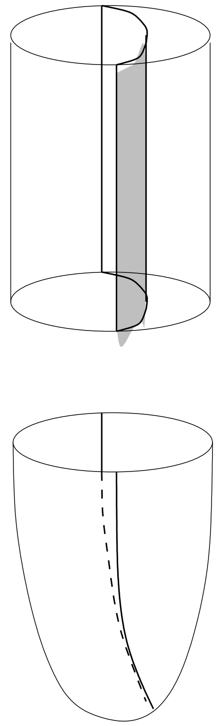

Figure 5: A D1-brane in the cigar descending from an AdS_2 brane in H_3 .

4.3 D1-branes from descent

To obtain the D1-branes on the cigar from the H_3 model, we follow the same strategy as in subsection 3.3, only that this time we start with the euclidean AdS_2 brane on H_3. Furthermore, when we insert the product formula (2.15) into the one-point function, we have to impose Dirichlet boundary conditions on the free boson. With such a choice, we obtain a 2-dimensional brane in the numerator theory which then descends to a D1-brane on the cigar.

We recall from [1] that the euclidean AdS_2 -branes are characterized by the following set of one-point functions

$$\begin{aligned} \langle \Phi^{\mathrm{H},j}_{np}(z,\bar{z})\rangle^{AdS_2}_r \; &= \delta(p)\,\mathcal{N}'(b)\,\frac{\Gamma(2j+1)}{\Gamma(1+j+n/2)\Gamma(1+j-n/2)}\,\times \\ &\times \left(e^{-r(2j+1)}+(-1)^n e^{r(2j+1)}\right)\,\Gamma(1+b^2(2j+1))\,\nu_b^{j+1/2}\,\frac{1}{|z-\bar{z}|^{2\Delta^{\mathrm{H}}_j}}\,. \end{aligned} \tag{4.15}$$

The pre-factor $\mathcal{N}'(b)$ is the same as in eq.~(4.7) We also need the one point functions of the U(1)vertex operators. In case of Dirichlet boundary conditions on the free boson they read

$$\langle V_{nw}(z,\bar{z})\rangle^D = \delta_{w,0}\,\frac{(2k)^{-1/4}}{|z-\bar{z}|^{n^2/2k}}\,, \tag{4.16}$$

Our expression (4.7) corresponds to taking the product of eqs. (4.15) and (4.16) with p being replaced by $-ikw$.

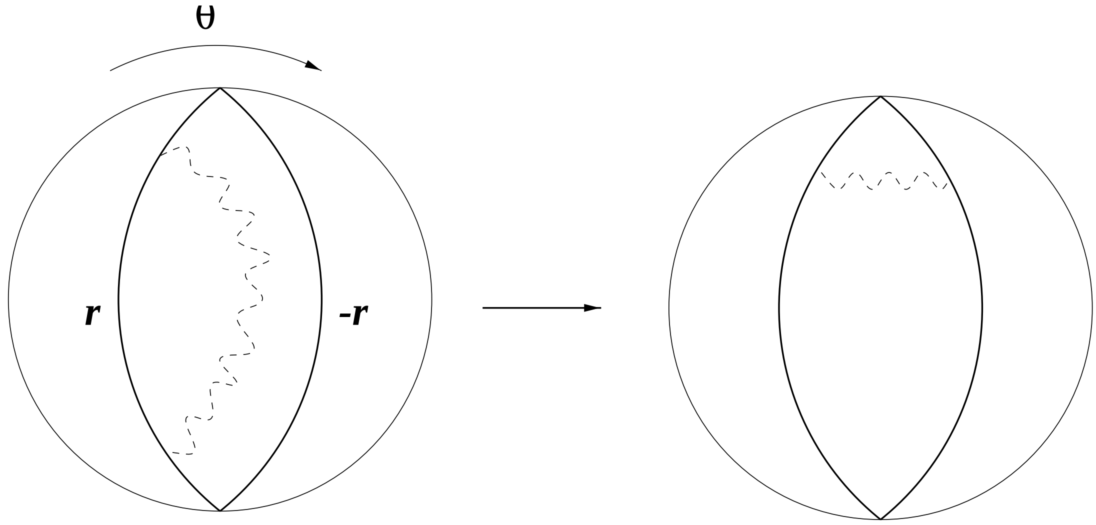

Figure 6: In AdS_3 viewed from above, the action of spectral flow on an open string ending on an AdS_2 brane.

The AdS_2 branes in H_3 extend along the time direction which we Wick-rotate and then gauge in order to obtain the cigar. Thus, in the terminology of [32], the D1-branes in the cigar should be thought of as non-compact A-branes (see figure 5). As in the compact case, the properties of A-branes are very easily deduced from those of parent branes. In particular, the open string density of states on a single brane is given by the same function $N(P|r,r)$.

Only the appearance of the winding number w in the open string spectrum makes the story a bit more subtle. In the context of AdS_2 branes in AdS_3 , we interpret $2w$ as counting the number of spectral flow operations. It is well known that for $r \neq 0$ only an even number of spectral flow operations preserves the spectrum of open strings living on an AdS_2 D-brane [35, 36]. If we apply an odd number of spectral flow operations to an open string with two ends on the same AdS_2 brane then one of its ends gets mapped to the opposite D-brane with parameter $-r$, see figure 6. These considerations explain why the spectral density for D1-branes on the cigar depends only on the parity of $2w$ and they identify the density of open strings with $w \in \mathbb{Z} + \frac{1}{2}$ with the quantity $N(P|r,-r)$ that was computed for branes in H_3 (see [18]). The same reasoning applies to the more complicated case of open strings stretched between two D1-branes with different parameters r, r'.

4.4 Cardy consistency condition

As in our discussion of the D0-branes we conclude the section in the D1-branes by showing that the exact solution we have proposed is consistent with world-sheet duality. Our presentation will be rather brief since most of it is rather similar to the case of AdS_2 branes in AdS_3 (see [1, 18]). In particular, we will restrict our computations to the case $r = r'$.

Our main aim is to show how the two contributions from $w \in \mathbb{Z}$ and $w \in \frac{1}{2} + \mathbb{Z}$ in the open string partition function (4.13) arise from world-sheet duality. To this end, we start from the closed string amplitude[7]

$$Z_{(r,\theta_0)(r',\theta_0)}(\tilde{q}) = \int dP \sum_{n\in\mathbb{Z}} \chi^c_{(P,n)}(\tilde{q})\, \Psi_{(r,\theta_0)}\left(-\frac{1}{2}+iP,n\right)\, \Psi_{(r',\theta_0)}\left(-\frac{1}{2}+iP,n\right)^* , \tag{4.17}$$

where

$$\begin{aligned}\Psi_{(r,\theta_0)}(j,n) \;=\; & (kb^2)^{-1/4}\, e^{in\theta_0} \frac{2\Gamma(2j+1)}{\Gamma(1+j+n/2)\Gamma(1+j-n/2)} \times \\ & \times \left(e^{-r(2j+1)} + (-1)^n e^{r(2j+1)}\right)\, \Gamma(1+b^2(2j+1))\, \nu_b^{j+1/2}\end{aligned} \tag{4.18}$$

are the couplings of closed strings to D1-branes which can be read off from eq.$\tilde{(}4.7)$. In our computation of this amplitude for the special case $r = r'$ we shall focus on the n-dependence since this is what distinguishes the calculation from the case of AdS_2 branes in H_3 . For the integrand of the amplitude Z we obtain

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \tilde{q}^{\frac{n^2}{4k}}\, \Psi_{(r,\theta_0)}\left(-\frac{1}{2}+iP,n\right)\, \Psi_{(r',\theta_0)}\left(-\frac{1}{2}+iP,n\right)^* \propto \\ &\qquad \propto \cosh^2 \pi P\ \cos^2 2rP \sum_{n\in 2\mathbb{Z}} \tilde{q}^{n^2/4k} + \sinh^2\pi P\ \sin^2 2rP \sum_{n\in 2\mathbb{Z}+1} \tilde{q}^{n^2/4k} .\end{aligned}$$

Here we used that the quantity $\Psi(.,n)\Psi(.,n)^*$ depends on n only through its parity. Nevertheless, the infinite sum over n is convergent thanks to n-dependent factor $\exp(-2\pi i n^2/4k\tau)$ that comes from the coset character (2.8). It is now straightforward to rewrite the previous quantity as

$$= (\cosh^2 \pi P\ \cos^2 2rP + \sinh^2\pi P\ \sin^2 2rP)\, \frac{1}{2} \sum_{n\in\mathbb{Z}} \tilde{q}^{\frac{n^2}{4k}} + \tag{4.19}$$

$$+ (\cosh^2 \pi P\ \cos^2 2rP - \sinh^2\pi P\ \sin^2 2rP)\, \frac{1}{2} \left(\sum_{n\in 2\mathbb{Z}} - \sum_{n\in 2\mathbb{Z}+1}\right) \tilde{q}^{n^2/4k} \tag{4.20}$$

$$\propto \cos^2 2rP \sum_{w\in\mathbb{Z}} q^{kw^2} + \cosh 2\pi P\ \cos^2 2rP \sum_{w\in\mathbb{Z}+1/2} q^{kw^2} + r - \mathrm{indep}. \tag{4.21}$$

The last step was a Poisson resummation which changed n into w. This shows how the two sectors with $w \in \mathbb{Z}$ and $w \in \mathbb{Z} + \frac{1}{2}$ appear in the open-string spectrum and that their densities differ by a factor $\cosh 2\pi P$. After modular transformation, the extra factor gives rise to the $\cosh t$ factor that we introduced in the definition of the special function (4.11) and that is not present in the corresponding quantity (4.10) for integer winding number w.

[7]To be precise, this quantity again diverges with the volume and to make it well defined, one should either divide by the volume divergence or subtract the same amplitude for a fixed reference brane $r = 0$ (see the discussion in [1]).

5. The D2-branes

After having discussed D0 and D1-branes we are left with one more species of branes. These are 2-dimensional objects that extend all the way to $\rho = \infty$. They can carry a 2-form gauge field F and hence are characterized by one real parameter. In the first subsection we shall argue for the existence of such branes using the Born-Infeld action. The explicit formulas we derive for the various open string background fields can be employed to derive a minisuperspace formula for the spectral density of open string states ending on the D2-branes. We present our exact solution for D2-branes in the second subsection before explaining how it may be obtained by descending from certain 2-dimensional branes in H_3 . Finally, we check the consistency of our proposal for the bulk one-point function and the open string spectral density with world-sheet duality. We shall show, in particular, how discrete open string modes on the D2-brane emerge within this Cardy computation.

5.1 Semi-classical description

The main feature that distinguishes the D2-branes from the branes we have discussed above is that they can carry a world-volume 2-form gauge field $F = F_{\rho\theta} d\rho \wedge d\theta$. In the presence of the latter, the Born-Infeld action for a D2-brane on the cigar becomes

$$S_{\rm BI} \propto \int d\rho d\theta \cosh\rho \sqrt{\tanh^2\rho + F_{\rho\theta}^2}\,. \tag{5.1}$$

We shall choose a gauge in which the component A_ρ of the gauge field vanishes so that we can write $F_{\rho\theta} = \partial_\rho A_\theta$. A short computation shows that the equation of motion for the one-form gauge potential A is equivalent to

$$F_{\rho\theta}^2 = \frac{\beta^2 \tanh^2\rho}{\cosh^2\rho - \beta^2}\,. \tag{5.2}$$

If the integration constant β is greater than one, then the D2-brane is localized in the region $\cosh\rho \geq \beta$, i.e. it does not reach the tip of the cigar. We will exclude this case in our semi-classical discussion and assume that $\beta = \sin\sigma \leq 1$. The corresponding D2-brane cover the whole cigar. Integrating eq.$\tilde{}$(5.2) for the field strength $F_{\rho\theta} = \partial_\rho A_\theta$ furnishes the following expression for the gauge potential

$$A_\theta(\rho) = \sigma - \arctan\left(\frac{\tan\sigma}{\sqrt{1+\frac{\sinh^2\rho}{\cos^2\sigma}}}\right)\,. \tag{5.3}$$

In our normalization $A_\theta(\rho = 0) = 0$, the parameter σ is the value of the gauge potential A_θ at infinity. When this parameter σ tends to $\sigma = \frac{\pi}{2}$, the F-field on the brane blows up. We should thus consider $\sigma = \frac{\pi}{2}$ as a physical bound for σ.

Let us point out that the F-field we found here vanishes at $\rho = \infty$. In other words, it is concentrated near the tip of the cigar. By the usual arguments, the presence of a non-vanishing F-field implies that our D2-branes carry a D0-brane charge which is given by the integral of the F-field. Like the F-field itself, the D0-brane charge is localized near the tip of the cigar, i.e. in a compact subset of the 2-dimensional background. Hence, one expects the D0-brane charge, and therefore the parameter σ of the D2-brane, to be quantized. Another way to argue for such a quantization is through semi-classical charge conservation [37]. Note that the difference of two D2-branes with parameters σ and σ' is a 2-sphere if we do not allow for deformations of the circle at infinity. This implies that the difference of the D2-brane parameters must be an integer m, [37]

$$\sigma - \sigma' = 2\pi \frac{m}{k}\,, \qquad m \in \mathbb{Z}\,. \tag{5.4}$$

We shall see later that a 'quantum corrected' version of this condition is needed in order to obtain a sensible spectrum of open strings between the two D2-branes.

Our next aim here is to predict the semi-classical limit of the open string spectrum on the D2-brane. This quantity can be obtained by analyzing the corresponding following *open-string laplacian*,

$$\Delta_{os} = -\frac{1}{e^{-2\Phi_{os}}\sqrt{\det G_{os}}}\partial_\mu e^{-2\Phi_{os}}\sqrt{\det G_{os}}G^{\mu\nu}_{os}\partial_\nu\,. \tag{5.5}$$

Note that Δ_{os} differs from the closed string laplacian Δ in that it involves the open string metric G_{os} and dilaton Φ_{os}. We can determine the latter from the explicit formulas for the metric g and the dilaton Φ of the cigar geometry. For the open string metric $G_{os} = g - Fg^{-1}F$ we find

$$ds^2_{os} = \frac{k}{2}\frac{1}{\cosh^2\rho - \sin^2\sigma}\left(\cosh^2\rho\, d\rho^2 + \sinh^2\rho\, d\theta^2\right). \tag{5.6}$$

Similarly, the open string dilaton is given by

$$e^{\Phi_{os}} = e^{\Phi}\sqrt{\det(g+F)g^{-1}} \;\propto\; \frac{1}{\sqrt{\cosh^2\rho - \sin^2\sigma}}\,. \tag{5.7}$$

Inserting both expressions into eq. (5.5), the open string laplacian takes the form

$$-\frac{k}{2}\Delta_{os} = \frac{2}{\sinh 2\rho}\,\partial_\rho \tanh\rho\,(\cosh^2\rho - \sin^2\sigma)\partial_\rho + \frac{\cosh^2\rho - \sin^2\sigma}{\sinh^2\rho}\,\partial_\theta^2\,. \tag{5.8}$$

The spectrum of this operator is the semi-classical limit of the open string spectrum on a D2-brane. As usual, the corresponding spectral density can be constructed from the reflection amplitude, i.e. from the behavior of eigen-functions under the reflection $j \to -j-1$. Let us only quote the results of this investigation: it turns out that the spectrum contains all values $\Delta^j = -j(j+1)/k + n^2/4k$ with $j = -\frac{1}{2} + iP$, and that the σ-dependence of the reflection amplitude is given by

$$R(j,n|\sigma,\sigma)_{k\to\infty} \;\propto\; (\cos^2\sigma)^{2j+1}\,. \tag{5.9}$$

This information suffices to find the relative spectral density of the system. We also note that, for obvious geometric reasons, we do not expect to find open string states with non-zero winding number in the spectrum.

5.2 The exact solution

Concerning the 2-dimensional branes we claim that there exists a 1-parameter family of exact solution. Just like the branes in our semi-classical discussion, the exact solutions are parametrized by a real parameter σ which now can take values in the interval

$$\sigma \in \Big[0 \, , \, \frac{\pi}{2} \left(1 + b^2\right) \Big[\, . \tag{5.10}$$

Note that this interval shrinks to its semi-classical analogue as we send b to zero. For the associated one-point functions of closed string modes in the presence of a D2-brane we propose

$$\begin{aligned} \langle \Phi^j_{nw}(z,\bar z)\rangle^{\rm D2}_\sigma \;=\; & \delta_{n,0}\, \mathcal{N}'(b) \left(\frac{\Gamma(-j+kw/2)}{\Gamma(j+1+kw/2)}\, e^{i\sigma(2j+1)} + \frac{\Gamma(-j-kw/2)}{\Gamma(j+1-kw/2)}\, e^{-i\sigma(2j+1)} \right) \times \\ & \times \left(\frac{k}{2}\right)^{1/4} \Gamma(2j+1)\,\Gamma(1+b^2(2j+1))\, \nu_b^{j+1/2}\, \frac{1}{|z-\bar z|^{h^j_{nw}+\bar h^j_{nw}}} \, , \end{aligned} \tag{5.11}$$

where $\mathcal{N}'(b) = (8b^2)^{-1/4}$ is the same factor as for D1-branes. This formula holds for closed string modes from the continuous series. It also encodes all information about the couplings of discrete modes, but they have to be read off carefully because of the infinite factors (see the discussion in the case of D0-branes).

The construction of the open string partition functions or rather of their spectral densities uses the same objects that we introduced in our discussion of the D1-branes. We are able to write down a consistent spectrum of open strings stretching between two D2-branes of parameters σ, σ' only if they satisfy the condition

$$\sigma - \sigma' = 2\pi \, \frac{m}{k-2} \, , \qquad m \in \mathbb{Z} \, . \tag{5.12}$$

This is a k-deformed version of the semi-classical condition (5.4). We claim that under the above condition, the open string spectrum between the branes σ and σ' contains the same discrete open string states as open string spectrum of one D0-brane of parameter m (see eq. (3.4)), and continuous states with relative spectral density

$$\Delta N(P,n|\sigma,\sigma') = \Delta N\left(P\,\middle|\, i\frac{\sigma+\sigma'}{2}, i\frac{\sigma+\sigma'}{2}\right) + \Delta N\left(P\,\middle|\, i\frac{\sigma-\sigma'}{2}, i\frac{\sigma'-\sigma}{2}\right) . \tag{5.13}$$

The relative density $\Delta N(P,n|\sigma,\sigma') = \Delta N(P,0|\sigma,\sigma')$ is the same as for open strings with $w \in \mathbb{Z}$ on a D1-brane but with the real brane parameters replaced by purely imaginary ones. In formulas we claim that

$$Z^{\rm D2}_{\sigma\sigma'}(q) = \int dP \sum_{n\in\mathbb{Z}} \Delta N(P,n|\sigma,\sigma')\, \chi^c_{(-\frac{1}{2}+iP,n)}(q) + Z^{\rm D0}_{mm}(q) \, , \tag{5.14}$$

where $m = (\sigma-\sigma')/2\pi b^2 \in \mathbb{Z}$. Note that the density of continuous open string states diverges when $\frac{\sigma+\sigma'}{2}$ reaches the upper bound $\frac{\pi}{2}(1+b^2)$ of eq. (5.10). Based on the explicit formulas for the open string spectral density we are now able to identify the parameter σ that enters the exact solution with the parameter σ in our semi-classical analysis. To this end we evaluate the exact formula for the spectral density in the semi-classical limit $b \to 0$ and compare the result with the semi-classical density obtained from eq. (5.9). The details are left to the reader.

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5.3 D2-branes from descent

We obtain the D2-branes by rotating the euclidean AdS_2 -branes in H_3 and then descending to the cigar (see figure 7). The relevant rotation matrix U can be found [1] and the corresponding one-point function are easily worked out,

$$\langle \Phi^{\mathrm{H},j}_{np}(z,\bar{z})\rangle_r = \mathcal{N}'(b)\,\delta_{n,0}\,\left[C'_j(p)\,e^{-r(2j+1)} + C'_j(-p)\,e^{r(2j+1)}\right]\,\frac{\nu_b^{j+1/2}\Gamma(1+b^2(2j+1))}{|z-\bar{z}|^{2\Delta_j}}\,, \tag{5.15}$$

where the function $C'_j(p)$ is defined through

$$C'_j(p) = \int_0^1 dy\; y^{-2j-1+ip}|y^2-1|^{2j} = \frac{\Gamma(2j+1)\Gamma(-j+ip/2)}{2\,\Gamma(j+1+ip/2)}\,.$$

Once more we multiply this by the one-point functions (3.7) of a free boson with Neumann boundary condition and insert $p = -ikw$ to obtain our formula (5.11) for the bulk one-point function of a D2-brane. The AdS_2 brane parameter r is related to the parameter σ in the 2-form field strength (5.2) on D2-branes via $r = i\sigma$. Let us mention that we could introduce a second parameter by shifting the branes in H_3 along the τ direction. This extra freedom is associated with turning on a Wilson line. Since it enters simply as w-dependent phase, we will not consider this any further.

The D2-branes on the cigar can be considered as non-compact analogues of the B-branes in the coset SU(2)/U(1) (*see* [32]). In fact, the U-rotated euclidean AdS_2 branes in H_3 from which we descended to our D2-branes may be Wick rotated into H_2 branes in

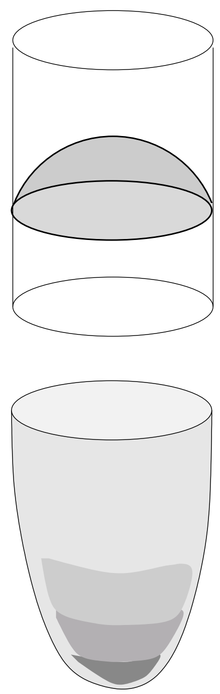

Figure 7: A D2-brane in the cigar descending from an H_2 brane in $\mathrm{SL}(2,\mathbb{R})$ or H_3 .

AdS_3 . The latter are localized along the surfaces

$$\cosh\rho\,\sin t = \sin\sigma\,, \tag{5.16}$$

i.e. along conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. Notice that D2-branes with $\cosh\rho \geq \beta$ which cover only part of the cigar should likewise be related to the unphysical dS_2 branes in AdS_3 [38].

5.4 Cardy consistency condition

Our last task here is to evaluate the annulus amplitude between two D2-branes of parameters σ and σ' from the boundary states (5.11) and to compare the answer with our expression (5.13) for the open string spectrum. Using the couplings Ψ_σ in the boundary states, we can compute

$$Z^{\mathrm{D2}}_{\sigma\sigma'}(\tilde{q}) = \int dj \sum_w \frac{\Psi_\sigma(j,w)\Psi_{\sigma'}(j,w)^*}{\left\langle \Phi^j_{0w}\Phi^j_{0w}\right\rangle}\, \chi_{(j,\frac{kw}{2})}(\tilde{q})\,. \tag{5.17}$$

As it stands, this is just a formal expression in which the 'integral' runs over both continuous and discrete series. The two-point function in the denominator is trivial in the case of the continuous series but is has a non-trivial dependence on the labels of discrete closed string states. We can spell out the integrand of eq. (5.17) explicitly, in a form that is valid for discrete and continuous values of the spin j,

$$\begin{aligned}\frac{\Psi_\sigma(j,w)\Psi_{\sigma'}(j,w)^*}{\left\langle \Phi^j_{0w}\Phi^j_{0w}\right\rangle} &= \frac{\pi^2\sqrt{kb^2}}{\sin 2\pi j\ \sin\pi b^2(2j+1)}\times\\ &\quad\times\left[2\cos(2j+1)(\sigma+\sigma') + \frac{\sin\pi(j+\frac{kw}{2})}{\sin\pi(j-\frac{kw}{2})}e^{-i(2j+1)(\sigma-\sigma')} + \right.\\ &\quad\left. + \frac{\sin\pi(j-\frac{kw}{2})}{\sin\pi(j+\frac{kw}{2})}e^{i(2j+1)(\sigma-\sigma')}\right]. \end{aligned}\tag{5.18}$$

In order to modular transform the annulus amplitude (5.17) we employ simple trigonometric identities and rewrite the previous formula for the coefficients as

$$\frac{\Psi_\sigma(j,w)\Psi_{\sigma'}(j,w)^*}{\left\langle \Phi^j_{0w}\Phi^j_{0w}\right\rangle} = \frac{2\pi^2\sqrt{kb^2}}{\sin\pi b^2(2j+1)}\ \{a_I + a_{II} + a_{III} + a_{IV}\}\,, \tag{5.19}$$

where the four terms in the brackets are given by

$$a_I = \frac{\cos(2j+1)(\sigma+\sigma')}{\sin 2\pi j} \tag{5.20}$$

$$a_{II} = \cos(2j+1)(\sigma-\sigma')\ \coth 2\pi j \tag{5.21}$$

$$a_{III} = \cos(2j+1)(\sigma-\sigma')\ \frac{\sin 2\pi j}{\cos 2\pi j - \cos\pi kw} \tag{5.22}$$

$$a_{IV} = i\ \sin(2j+1)(\sigma-\sigma')\ \frac{\sin\pi kw}{\cos 2\pi j - \cos\pi kw}\,. \tag{5.23}$$

Let us note that the fourth term a_{IV} does not contribute to the annulus amplitude because it is odd in w. The three other terms, however, are real and non-vanishing. Contributions from the discrete closed string states are encoded entirely in the third term a_{III}. In fact, discrete states can only contribute through singularities of the integrand (see subsection 3.4) and the terms a_I and a_{II} are finite for physical values of j in the discrete series.[8] The term a_{III}, on the other hand, is infinite when (j, w) fall into the set of physical discrete states. The associated contribution to the annulus amplitude is given by the residue of the pole. With this in mind it is not hard to see that the term a_{III} furnishes both the continuous and the discrete closed string modes' contributions to the annulus amplitude of a D0-brane with parameter $m = \frac{k-2}{2\pi}(\sigma - \sigma')$, provided this m is an integer (see equations (3.4) and (3.19). We thus recover the announced discrete part of open string spectrum on D2-branes along with a quantization condition for the difference $\sigma - \sigma'$ of the branes' parameter. When the latter is not obeyed, we cannot interpret our amplitude as an open string partition function.

We still have to deal with terms a_I and a_{II}. Their contribution comes entirely from continuous closed string states and it is rather easy to modular transform. Indeed, the integral over P and the sum over w decouple and we find continuous open string modes with density (5.13). Note that the computation gives a well-defined density only if the mean $(\sigma + \sigma')/2$ of the branes' parameters σ and σ' belongs to the interval (5.10). In particular, the parameter of any single brane has to belong to the same interval so that we obtain an exact version of the classical bound on σ. This concludes the Cardy check for D2-branes in the cigar.

6. Conclusion and open issues

In this work we have presented a complete analysis of (maximally symmetric) branes in the $2D$ black hole background. The main results include exact formulas (3.3), (4.7), (5.11) for the various boundary states and the boundary partition functions (3.4), (4.13), (5.14). To write down the latter we have also presented formulas for the boundary reflection amplitudes, i.e. for the 2-point functions of boundary fields. It would certainly be worthwhile working out formulas for the spectrum of open strings that stretch in between branes of different dimension. At the moment we only have expressions for open strings stretching between D0 and D2-branes.

The most important information, however, that is missing for a complete solution of the boundary theories are the various boundary 3-point functions. Unfortunately, these are not known even for the H_3 theory and so finding explicit formulas for the 3-point couplings remains an interesting open problem for future research. Its solution would be a crucial step toward studying the dynamics of extended branes on the cigar (see [39] for related work on Liouville theory).

[8] This may not always be true when k is rational, in which case half-integer values of j may appear in the spectrum. However, the positions of the corresponding poles of a_I and a_{II} are then independent of w, and we believe that they should not be taken into account.

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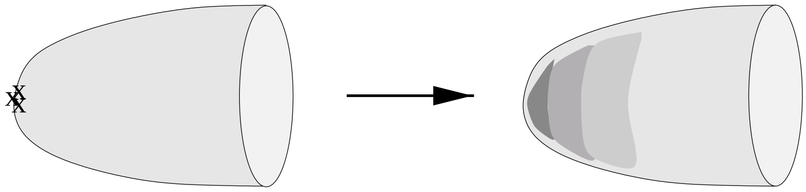

Figure 8: D0-branes being absorbed by a D2-brane.

Here we can only formulate two conjectures on brane dynamics. The first one concerns systems of D0-branes and a D2-brane (see also [37]). Given the identification of the D2-brane parameter σ with a D0-brane charge (see last section) it is tempting to conjecture that the bound state between N D0-branes with $m = 1$ and a single D2-brane of parameter σ results in a single D2-brane with parameter $\sigma' = \sigma + 2\pi b^2 N$. In other words, we imagine a process in which a D2-brane absorbs a single D0-brane or a finite number thereof, as in figure 8. Note that our identification of the bound state assumes that the parameters stay in the 'physical' range $\sigma + 2\pi b^2 N < \pi(1 + b^2)/2$ and we do not have any candidate boundary state for the condensate if this bound is exceeded.

Another dynamical process occurs when we put a pair of D2-branes on the cigar with parameters σ_1, σ_2 satisfying $|\sigma_1 - \sigma_2| > 2\pi b^2$. We believe that such a system will decay into a stable system of two D2-branes whose parameters σ_1' and σ_2' differ at most by one, $|\sigma_1' - \sigma_2'| \leq 2\pi b^2$. This can be achieved through an exchange of D0-brane charge between the two D2-branes. It would certainly be interesting to verify these two proposals through a perturbative analysis in the exact boundary CFT (see [34, 40] for similar studies in the case of compact parafermions).

There exist several other issues that deserve further investigation. The most interesting involve the D0-branes at the tip of the cigar. We have pointed out at several places throughout the paper that the D0 and D2-branes are non-compact analogues of the B-branes in [32]. In the compact case, B-branes have been used to construct symmetry breaking branes on the group manifold SU(2) . These are either 1- or 3-dimensional. A similar construction (see [41] for a more systematic treatment) can be applied to the D0-branes in the $2D$ black hole and it provides us with a brane that stretches out along the 1-dimensional line $\rho = 0$ in H_3 . The existence of such symmetry-breaking branes in $SL(2, \mathbb{R})$ can also be inferred from a semi-classical analysis [42, 43], but their exact construction was not known before (see [44] for some attempts in this direction). Volume filling branes in H_3 can be obtained similarly from the D2-branes in the black hole.

Another direction involving primarily the D0-branes concerns the generalization of recent developments in 2-dimensional string theory and the $c = 1$ matrix model. It was argued in [45]–[47] that the well known matrix dual of Liouville theory may be interpreted as the effective field theory describing the dynamics of localized branes in Liouville field theory. The dual matrix model for the $2D$ black hole background has been found in [13] and, in complete analogy to the Liouville case, it should describe the dynamics of our D0-branes [48]. Higher dimensional generalizations of this duality, which all contain the $2D$

black hole as a building block, arise in the context of little string theory. The results we have described above allow to construct various branes in the dual 9+1-dimensional string backgrounds.

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A. Characters of discrete representations

In this appendix we derive the formula (2.8) that we use for the characters of the discrete series on the cigar. Our strategy is to insert the expression (2.18) for characters of discrete $SL(2,\mathbb{R})$ representations into relation (2.19) and to evaluate the integral explicitly. Before we go into this computation, let us make a few comments on the function $\vartheta_1(q,z)$. It is defined by

$$\begin{aligned}\vartheta_1(q = e^{2\pi i\tau}, z = e^{2\pi i\theta}) &= -2q^{1/8}\sin\pi\theta \prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^n) \\ &= \sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}(m-\frac{1}{2})^2} z^{m-\frac{1}{2}}\end{aligned}$$

and enjoys the following behavior under modular transformation

$$\vartheta_1(q,z) = -(-i\tau)^{-1/2}\, e^{-i\pi\theta^2/\tau}\, \vartheta_1(\tilde{q},\tilde{z})$$

where $\tilde{q} = \exp(-2\pi i/\tau)$ and $\tilde{\theta} = \theta/\tau$.

The computation of the integral (2.19), and therefore the proof of formula (2.8), is easily seen to boil down to the calculation of the following contour integrals for $\kappa \in \mathbb{Z}$,

$$\frac{1}{2\pi i}\int \frac{dz}{z}\frac{z^{\kappa}}{1-z}\frac{1}{\Pi_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^n)} \tag{A.1}$$

over the circle $|z| = 1-\epsilon$ (recall that we displaced the contour in eq.$\tilde{(}$2.19) slightly to regularize the integral). If $\kappa \geq 1$ then all singularities of the integrand inside the circle come from the first order poles at $z = q^m$. Hence, the contour integral can be evaluated using that in the vicinity of $z = q^m$ we have

$$\frac{1}{1-z}\frac{1}{\Pi_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^n)} \simeq \frac{q^m}{z-q^m}\frac{q^{1/12}}{\eta(q)^2}(-1)^{m+1}q^{m(m-1)/2}\,. \tag{A.2}$$

If $\kappa \leq 0$, on the other hand, we move the integration contour to infinity, picking up contributions from the poles at $z = 1$ and $z = q^{-m}$, all decorated with an extra minus sign to take the orientation of the contour into account. The results of these manipulations agree with our expressions (2.8) for the discrete coset characters.

B. Cardy computation for S^2 -branes in H_3

Our aim here is to extend the proof of world-sheet duality for the S^2 -branes in H_3 (see [1]). In contrast to the calculations performed in [1], we will not only check the world-sheet duality for the usual annulus amplitude, but for the more interesting quantity,

$$Z^{S^2}_{mm'}(q,z) = \mathrm{Tr} q^{L_0 - c/24}\ z^{J^0_0}\,. \tag{B.1}$$

The operator J^0_0 denotes the zero mode (in the sense of affine Lie algebras) of the $\mathrm{SL}(2,\mathbb{C})$ current J^0 of the H_3 sigma model. In our analysis we compute the quantity $Z^{S^2}_{mm'}(q,z)$ from the one-point functions (3.6) of the S^2 branes in H_3 and express the result in terms of unspecialized characters,

$$\begin{aligned}
\tilde{q}^{\frac{k}{4}\theta^2}\ Z^{S^2}_{mm'}(q,z) &= {}^{S^2}\langle m|\ \tilde{q}^{(L_0+\bar{L}_0)-\frac{c}{24}}\ e^{2\pi i\tilde{\theta}J^0_0}\ |m'\rangle^{S^2} \\
&= \int dPdp\ \Psi^{S^2}_{m}\left(-\frac{1}{2}+iP\right)\ \Psi^{S^2}_{m'}\left(-\frac{1}{2}+iP\right)^*\ \tilde{q}^{b^2P^2}\ e^{-\pi\tilde{\theta}p}\ \frac{(-2)\sin\pi\tilde{\theta}}{\vartheta_1(\tilde{q},\tilde{z})} \\
&= -2\sqrt{2b^2}\int dP\ \frac{\tilde{q}^{b^2P^2}}{\vartheta_1(\tilde{q},\tilde{z})}\sum_J \sinh 2\pi b^2 P(2J+1)\ \sinh 2\pi P\tilde{\theta}\,,
\end{aligned} \tag{B.2}$$

where have inserted the eigen-values $i\frac{p}{2}$ of the operator J^0_0 and $\Psi^{S^2}_m(j) = \langle\Phi^{\mathrm{H},j}_{np}(z=\frac{i}{2})\rangle_m$. In the last line the summation over J runs over the same set as in eq.(3.4). Now we are prepared to perform the modular transformation,

$$Z^{S^2}_{mm'}(q,z) = -2\sum_J q^{-b^2(J+\frac{1}{2})^2}\ \frac{\sin\pi\theta(2J+1)}{\vartheta_1(q,z)} \tag{B.3}$$

$$= \sum_J \left(\chi^{\mathrm{H},d}_J(q,z) - \chi^{\mathrm{H},d}_{-J-1}(q,z)\right). \tag{B.4}$$

We thus find the same discrete spectrum as in [1], now written in terms of unspecialized characters (2.18).

References

[1] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the euclidean AdS_3*, *J. High Energy Phys.* **02** (2002) 016 [hep-th/0112198].

[2] S. Elitzur, A. Forge and E. Rabinovici, *Some global aspects of string compactifications*, *Nucl. Phys.* **B 359** (1991) 581.

[3] G. Mandal, A.M. Sengupta and S.R. Wadia, *Classical solutions of two-dimensional string theory*, *Mod. Phys. Lett.* **A 6** (1991) 1685.

[4] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314.

[5] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys.* **B 360** (1991) 197.

[6] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164].

[7] K. Sfetsos, *Branes for Higgs phases and exact conformal field theories*, *J. High Energy Phys.* **01** (1999) 015 [hep-th/9811167].

[8] D. Kutasov, *Introduction to little string theory*, prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2–10 Apr 2001.

[9] R. Dijkgraaf, H. Verlinde and E. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B 371** (1992) 269.

[10] A. Hanany, N. Prezas and J. Troost, *The partition function of the two-dimensional black hole conformal field theory*, *J. High Energy Phys.* **04** (2002) 014 [hep-th/0202129].

[11] J. Teschner, *Operator product expansion and factorization in the H_3^+ WZNW model*, *Nucl. Phys.* **B 571** (2000) 555 [hep-th/9906215].

[12] J. Teschner, *On structure constants and fusion rules in the* $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 546** (1999) 390 [hep-th/9712256].

[13] V. Kazakov, I.K. Kostov and D. Kutasov, *A matrix model for the two-dimensional black hole*, *Nucl. Phys.* **B 622** (2002) 141 [hep-th/0101011].

[14] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *J. High Energy Phys.* **10** (1999) 034 [hep-th/9909110].

[15] K. Hori and A. Kapustin, *Duality of the fermionic 2D black hole and $N = 2$ Liouville theory as mirror symmetry*, *J. High Energy Phys.* **08** (2001) 045 [hep-th/0104202].

[16] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary liouville field theory, I. Boundary state and boundary two-point function*, hep-th/0001012.

[17] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[18] S. Ribault, *Two AdS_2 branes in the euclidean AdS_3*, *J. High Energy Phys.* **05** (2003) 003 [hep-th/0210248].

[19] A. Giveon, D. Kutasov and A. Schwimmer, *Comments on D-branes in AdS_3*, *Nucl. Phys.* **B 615** (2001) 133 [hep-th/0106005].

[20] A. Rajaraman and M. Rozali, *Boundary states for D-branes in AdS_3*, *Phys. Rev.* **D 66** (2002) 026006 [hep-th/0108001].

[21] P. Lee, H. Ooguri and J.W. Park, *Boundary states for AdS_2 branes in AdS_3*, *Nucl. Phys.* **B 632** (2002) 283 [hep-th/0112188].

[22] A. Fotopoulos, *Semiclassical description of D-branes in* $\mathrm{SL}(2)/\mathrm{U}(1)$ *gauged WZW model*, *Class. Quant. Grav.* **20** (2003) S465–S472 [hep-th/0304015].

[23] S. Ribault, *Strings and D-branes in curved space-times. (in french)*, hep-th/0309272.

[24] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and* $\mathrm{SL}(2,\mathbb{R})$ *WZW model, I*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053].

[25] A. Pakman, *Brst quantization of string theory in AdS_3*, *J. High Energy Phys.* **06** (2003) 053 [hep-th/0304230].

[26] K. Becker and M. Becker, *Interactions in the* $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ *black hole background*, *Nucl. Phys.* **B 418** (1994) 206 [hep-th/9310046].

[27] G. Giribet and C. Núñez, *Correlators in AdS_3 string theory*, *J. High Energy Phys.* **06** (2001) 010 [hep-th/0105200].

[28] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and the* $\mathrm{SL}(2,\mathbb{R})$ *WZW model, III. Correlation functions*, *Phys. Rev.* **D 65** (2002) 106006 [hep-th/0111180].

[29] J. Teschner, *The mini-superspace limit of the* $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 546** (1999) 369 [hep-th/9712258].

[30] J. Teschner, *Crossing symmetry in the H_3^+ WZNW model*, *Phys. Lett.* **B 521** (2001) 127 [hep-th/0108121].

[31] B. Ponsot, *Monodromy of solutions of the Knizhnik-Zamolodchikov equation:* $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 642** (2002) 114 [hep-th/0204085].

[32] J.M. Maldacena, G.W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *J. High Energy Phys.* **07** (2001) 046 [hep-th/0105038].

[33] K. Gawędzki, *Boundary WZW, G/H, G/G and CS theories*, *Annales Henri Poincaré* **3** (2002) 847–881 [hep-th/0108044].

[34] S. Fredenhagen and V. Schomerus, *D-branes in coset models*, *J. High Energy Phys.* **02** (2002) 005 [hep-th/0111189].

[35] P.M. Petropoulos and S. Ribault, *Some remarks on anti-de Sitter D-branes*, *J. High Energy Phys.* **07** (2001) 036 [hep-th/0105252].

[36] P. Lee, H. Ooguri, J. w. Park and J. Tannenhauser, *Open strings on AdS_2 branes*, *Nucl. Phys.* **B 610** (2001) 3 [hep-th/0106129].

[37] C. Bachas, A. Fotopoulos, M. Petropoulos and S. Ribault, unpublished 2003.

[38] C. Bachas and M. Petropoulos, *Anti-de-sitter D-branes*, *J. High Energy Phys.* **02** (2001) 025 [hep-th/0012234].

[39] J. Teschner, *On boundary perturbations in Liouville theory and brane dynamics in noncritical string theories*, hep-th/0308140.

[40] S. Fredenhagen, *Organizing boundary RG flows*, *Nucl. Phys.* **B 660** (2003) 436 [hep-th/0301229].

[41] T. Quella and V. Schomerus, *Symmetry breaking boundary states and defect lines*, *J. High Energy Phys.* **06** (2002) 028 [hep-th/0203161].

[42] G. Sarkissian, *Non-maximally symmetric D-branes on group manifold in the lagrangian approach*, *J. High Energy Phys.* **07** (2002) 033 [hep-th/0205097].

[43] T. Quella, *On the hierarchy of symmetry breaking D-branes in group manifolds*, *J. High Energy Phys.* **12** (2002) 009 [hep-th/0209157].

[44] A. Rajaraman, *New AdS_3 branes and boundary states*, hep-th/0109200.

[45] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *J. High Energy Phys.* **12** (2003) 054 [hep-th/0304224].

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[46] E.J. Martinec, *The annular report on non-critical string theory*, hep-th/0305148.

[47] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *J. High Energy Phys.* **07** (2003) 045 [hep-th/0305159].

[48] S.Y. Alexandrov, V.A. Kazakov and D. Kutasov, *Non-perturbative effects in matrix models and D-branes*, *J. High Energy Phys.* **09** (2003) 057 [hep-th/0306177].